

A generalized Benamou-Brenier formula for mass-varying densities

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Abstract

In this article, we present a generalization of the Benamou-Brenier formula, that links the Wasserstein distance to an action functional related to the transport equation. On one side, we introduce a generalization of the Wasserstein distance for mass-varying measures. On the other side, we define an action functional related to the transport equation with source.

Keywords: transport equation, evolution of measures, Wasserstein distance.

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1 Introduction

The problem of optimal transportation, also called Monge-Kantorovich problem, has been intensively studied in the mathematical community. Related to such problem, the definition of the Wasserstein distance in the space of probability measure has revealed itself to be a powerful tool, in particular for dealing with dynamics of measures (like the transport PDE, see e.g. [1]). For a complete introduction to Wasserstein distances, see [6, 7].

The main limit of this approach, at least for its application to dynamics of measures, is that the Wasserstein distance $W_p(\mu, \nu)$ is defined only if the two measures μ, ν have the same mass. For this reason, in [5] we defined a generalized Wasserstein distance $W_p^{a,b}(\mu, \nu)$, combining the standard Wasserstein and L^1 distances. In rough words, for $W_p^{a,b}(\mu, \nu)$ an infinitesimal mass $\delta\mu$ of μ (or ν) can either be removed at cost $a|\delta\mu|$, or moved from μ to ν at cost $bW_p(\delta\mu, \delta\nu)$. This distance is a generalization of the “flat distance”.

One interesting field of application of this new distance is the study of transport PDEs with source, i.e. the following dynamics of measures:

$$\partial_t \mu_t + \nabla \cdot (v_t \mu_t) = h_t \quad (1)$$

Several authors have studied (1) without source term, i.e. $h \equiv 0$, showing that it is very convenient to use the standard Wasserstein distance in this framework. In particular, Benamou and Brenier showed in [2] that there is a natural equivalence between the minimization of the action functional $\mathcal{A}[\mu, \nu] := \int_0^1 dt \left(\int_{\mathbb{R}^d} d\mu_t |v_t|^2 \right)$ and the computation of the Wasserstein distance. Their fundamental result is recalled in Section 3.

The main limit of the approach based on the standard Wasserstein distance is that, as explained above, it cannot encompass the case of a source h . Indeed, in this case the mass of the measure μ_t varies in time, hence in general $W_p(\mu_t, \mu_s)$ is not defined for $t \neq s$. In this article, to deal with the source in (1), we focus on the space of Borel measures with finite mass on \mathbb{R}^d (denoted with \mathcal{M}), that we endow with the generalized

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Wasserstein distance $W_2^{a,b}$. We also denote with \mathcal{M}_0^{ac} the subspace of \mathcal{M} of measures that are absolutely continuous with respect to the Lebesgue measure and with bounded support.

Our goal here is to generalize the Benamou-Brenier formula to this setting. On one side, we define the generalized Wasserstein distance¹, mixing creation/removal of mass and transport of mass. We first define

$$T_2^{a,b}(\mu, \nu) = \inf_{\tilde{\mu}, \tilde{\nu} \in \mathcal{M}, |\tilde{\mu}|=|\tilde{\nu}|} a^2 (|\mu - \tilde{\mu}| + |\nu - \tilde{\nu}|)^2 + b^2 W_2^2(\tilde{\mu}, \tilde{\nu}),$$

from which we have the following definition of the generalized Wasserstein distance:

$$W_2^{a,b}(\mu, \nu) := \sqrt{T_2^{a,b}(\mu, \nu)}.$$

On the other side, we define a generalization of the functional \mathcal{A} , taking into account both the transport and the creation/removal of mass in (1). More precisely, we define

$$\mathcal{B}^{a,b}[\mu, v, h] := a^2 \left(\int_0^1 dt \left(\int_{\mathbb{R}^d} d|h_t| \right) \right)^2 + b^2 \int_0^1 dt \left(\int_{\mathbb{R}^d} d\mu_t |v_t|^2 \right).$$

Given the generalizations both for the distance and the functional, we will then prove the generalized Benamou-Brenier formula under the regularity hypotheses recalled in Definition 8:

$$T_2^{a,b}(\mu_0, \mu_1) = \inf \left\{ \mathcal{B}^{a,b}[\mu, v, h] \mid \begin{array}{l} \mu \text{ is a solution of (1) with vector field } v, \text{ source } h \\ \text{and } \mu|_{t=0} = \mu_0, \mu|_{t=1} = \mu_1 \end{array} \right\}. \quad (2)$$

The proof is based on two steps. We first prove (2) under some regularity assumptions for v, h , and approximating solutions of (1) with a sample-and-hold method. We then prove (2) in a more general setting, via a regularization argument.

The structure of the paper is the following. In Section 2 we define the generalized Wasserstein distance and prove some useful properties. In Section 3 we recall the standard Benamou-Brenier formula and state the main result of the paper, that is the generalized Benamou-Brenier formula. Section 4 is devoted to the study of solutions of (1) under strong regularity hypotheses for v, h and to the proof of (2) in this setting. Finally, Section 5 contains the proof of the generalized Benamou-Brenier in the general setting, that is based on a regularization argument.

2 Generalized Wasserstein distance

In this section we define the generalized Wasserstein distance and prove some useful properties. Several other results, like the characterization of the topology induced by such distance and the comparison with other distances, are presented in [5].

2.1 Notation and standard Wasserstein distance

We first fix the notation that we use throughout the paper, and recall definitions and properties related to measure theory and the Wasserstein distance, like push-forward of measures $\gamma \# \mu$ and transference plans.

Let μ be a positive Borel measure with locally finite mass. If μ_1 is absolutely continuous with respect to μ , we write $\mu_1 \ll \mu$. If $\mu_1(A) \leq \mu(A)$ for all Borel sets, we write $\mu_1 \leq \mu$. Given a measure with finite mass, we denote with $|\mu| := \mu(\mathbb{R}^d)$ its norm. More in general, if $\mu = \mu^+ - \mu^-$ is a signed Borel measure, we define $|\mu| := |\mu^+| + |\mu^-|$. Such norm defines a distance in \mathcal{M} , that is $|\mu - \nu|$. It is useful to recall that, if $\mu_1 \ll \mu$ and $d\mu_1 = f d\mu$ with $f \in L^1(d\mu)$, then $|\mu_1| = \int |f| d\mu$.

¹Observe that the definition in [5] was $W_2^{a,b}(\mu, \nu) = \inf_{\tilde{\mu}, \tilde{\nu} \in \mathcal{M}} a|\mu - \tilde{\mu}| + a|\nu - \tilde{\nu}| + bW_p(\tilde{\mu}, \tilde{\nu})$. Clearly, the two definitions are extremely similar, and satisfy similar properties: one can indeed observe that, given the vector $(a|\mu - \tilde{\mu}| + a|\nu - \tilde{\nu}|, bW_2(\tilde{\mu}, \tilde{\nu})) \in \mathbb{R}^2$, the definition in [5] is the 1-norm of such vector, while the definition given in the present article is its 2-norm.

Given two measures μ, ν , one can always write in a unique way $\mu = \mu_{ac} + \mu_s$ such that $\mu_{ac} \ll \nu$ and $\mu_s \perp \nu$, i.e. there exists B such that $\mu_s(B) = 0$ and $\nu(\mathbb{R}^n \setminus B) = 0$. This is the Lebesgue's decomposition Theorem. Then, it exists a unique $f \in L^1(d\nu)$ such that $d\mu_{ac}(x) = f(x) d\nu(x)$. Such function is called the Radon-Nikodym derivative of μ with respect to ν . We denote it with $D_\nu \mu$. For more details, see e.g. [3].

Given a Borel map $\gamma : \mathbb{R}^d \rightarrow \mathbb{R}^d$, one can consider the following action on a measure $\mu \in \mathcal{M}$:

$$\gamma \# \mu(A) := \mu(\gamma^{-1}(A)).$$

An evident property is that the mass of μ , i.e. $\mu(\mathbb{R}^d)$ is identical to the mass of $\gamma \# \mu$.

Given two measures μ, ν with the same mass, it is thus possible to ask if there exists a γ such that $\nu = \gamma \# \mu$. We say that γ sends μ to ν . Moreover, one can add a cost to such γ , given by $I[\gamma] := |\mu|^{-1} \int_{\mathbb{R}^d} |x - \gamma(x)|^p d\mu(x)$. This means that each infinitesimal mass $\delta\mu$ is sent to $\delta\nu$ and that its infinitesimal cost is related to the p -th power of the distance between them. The, one can consider the map γ minimizing such cost, if it exists. This is known as the Monge problem, stated by Monge in 1791.

In general, this procedure works only for special μ, ν and p . Indeed, there exist simple examples of μ, ν for which a γ that sends μ to ν does not exist. For example $\mu = 2\delta_1$, $\nu = \delta_0 + \delta_2$ on the real line have the same mass, but there exists no γ with $\nu = \gamma \# \mu$, since γ cannot separate masses. Moreover, one can have a sequence γ_n of maps such that $I[\gamma_n]$ is a minimizing sequence, but the limit is not a map γ^* . A simple condition that ensures the existence of a minimizing γ is that μ and ν are absolutely continuous with respect to the Lebesgue measure.

For such reason, one can generalize the problem to the following setting. Given a probability measure π on $\mathbb{R}^d \times \mathbb{R}^d$, one can interpret it as a method to transfer a measure μ on \mathbb{R}^d to another measure on \mathbb{R}^d as follows: each infinitesimal mass on a location x is sent to a location y with a probability given by $\pi(x, y)$. Formally, μ is sent to ν if the following properties hold:

$$|\mu| \int_{\mathbb{R}^d} d\pi(x, \cdot) = d\mu(x), \quad |\nu| \int_{\mathbb{R}^d} d\pi(\cdot, y) = d\nu(y). \quad (3)$$

Such π is called a transference plan from μ to ν . We denote the set of such transference plans as $\Pi(\mu, \nu)$. Since one usually deals with probability measures μ, ν , the terms $|\mu|, |\nu|$ are usually neglected in the literature. A condition equivalent to (3) is that, for all $f, g \in C_c^\infty(\mathbb{R}^d)$ it holds $|\mu| \int_{\mathbb{R}^d \times \mathbb{R}^d} (f(x) + g(y)) d\pi(x, y) = \int_{\mathbb{R}^d} f(x) d\mu(x) + \int_{\mathbb{R}^d} g(y) d\nu(y)$. Then, one can define a cost for π as follows $J[\pi] := \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\pi(x, y)$ and look for a minimizer of J in $\Pi(\mu, \nu)$. Such problem is called the Monge-Kantorovich problem.

It is important to observe that such problem is a generalization of the Monge problem. Indeed, given a γ sending μ to ν , one can define a transference plan $\pi = (\text{Id} \times \gamma) \# \mu$, i.e. $d\pi(x, y) = \mu(\mathbb{R}^n)^{-1} d\mu(x) \delta_{y=\gamma(x)}$. It also holds $J[\text{Id} \times \gamma] = I[\gamma]$. The main advantages of such approach are the following: first, the existence of at least one π satisfying (3) is easy to check, since one can choose $\pi(A \times B) = |\mu|^{-1} \mu(A) \nu(B)$, i.e. the mass from μ is proportionally split to ν . Moreover, a minimizer of J in $\Pi(\mu, \nu)$ always exists.

A natural space in which J is finite is the space of Borel measures with finite p -moment, that is

$$\mathcal{M}^p := \left\{ \mu \in \mathcal{M} \mid \int |x|^p d\mu(x) < \infty \right\}.$$

One can thus define on \mathcal{M}^p the following operator between measures of the same mass, called the **Wasserstein distance**:

$$W_p(\mu, \nu) = (|\mu| \min_{\pi \in \Pi(\mu, \nu)} J[\pi])^{1/p}.$$

It is indeed a distance on the subspace of measures in \mathcal{M}^p with a given mass, see [7]. It is easy to prove that $W_p(k\mu, k\nu) = k^{1/p} W_p(\mu, \nu)$ for $k \geq 0$, by observing that $\Pi(k\mu, k\nu) = \Pi(\mu, \nu)$ and that $J[\pi]$ does not depend on the mass. We will recall some other properties all along the paper, when useful.

2.2 Definition of the generalized Wasserstein distance

In this section, we recall the definition of the generalized Wasserstein distance and prove some useful properties.

Definition 1 Let $\mu, \nu \in \mathcal{M}$ be two measures. We define the functionals

$$T_2^{a,b}(\mu, \nu) := \inf_{\tilde{\mu}, \tilde{\nu} \in \mathcal{M}, |\tilde{\mu}|=|\tilde{\nu}|} a^2 (|\mu - \tilde{\mu}| + |\nu - \tilde{\nu}|)^2 + b^2 W_2^2(\tilde{\mu}, \tilde{\nu}), \quad (4)$$

and

$$W_2^{a,b}(\mu, \nu) := \sqrt{T_2^{a,b}(\mu, \nu)}. \quad (5)$$

In the following, we will also use the C^0 -norm for time-varying measures $\mu, \nu \in C([0, 1], \mathcal{M})$, that is

$$d(\mu, \nu) := \sup_{t \in [0, 1]} W_2^{a,b}(\mu_t, \nu_t). \quad (6)$$

We now recall some properties of $W_2^{a,b}$ and $T_2^{a,b}$. We give sketch of the proofs. Detailed proofs can be adapted from proofs given in [5].

Proposition 2 The following properties hold:

1. The infimum in (4) coincides with

$$\inf_{\tilde{\mu} \leq \mu, \tilde{\nu} \leq \nu, |\tilde{\mu}|=|\tilde{\nu}|} a^2 (|\mu - \tilde{\mu}| + |\nu - \tilde{\nu}|)^2 + b^2 W_2^2(\tilde{\mu}, \tilde{\nu}),$$

where we have added the constraint $\tilde{\mu} \leq \mu, \tilde{\nu} \leq \nu$.

2. The infimum in (4) is attained by some $\tilde{\mu}, \tilde{\nu}$.

3. The functional $W_2^{a,b}$ is a distance on \mathcal{M} .

4. It holds $W_2^{a,b}(\mu, 0) \leq a|\mu|$

Proof. Property 1. Take a pair $\tilde{\mu}, \tilde{\nu} \in \mathcal{M}_0^{ac}$ with $|\tilde{\mu}| = |\tilde{\nu}|$, and π the transference plan realizing $W_2(\tilde{\mu}, \tilde{\nu})$. Assume that $\tilde{\mu} \not\leq \mu$. Define $\mu' := \min\{1, D_{\tilde{\mu}}\mu\}$ and observe that, by construction $\mu' \leq \mu$. Also define ν' the measure such that π is a transference plan in $\Pi(\mu', \nu')$. Then, one can prove that

$$|\mu - \mu'| + |\nu - \nu'| \leq |\mu - \tilde{\mu}| + |\nu - \tilde{\nu}|, \quad W_2^2(\mu', \nu') \leq W_2^2(\tilde{\mu}, \tilde{\nu}).$$

If $\nu' \leq \nu$, then the result is proven. Otherwise, applying to ν' the techniques described above for $\tilde{\mu}$, one can define $\nu'' \leq \nu$ and a corresponding $\mu'' \leq \mu' \leq \mu$ such that $|\mu - \mu''| + |\nu - \nu''| \leq |\mu - \tilde{\mu}| + |\nu - \tilde{\nu}|$ and $W_2^2(\mu'', \nu'') \leq W_2^2(\tilde{\mu}, \tilde{\nu})$. The result easily follows.

Property 2. First use Property 1 to add the constraint $\tilde{\mu} \leq \mu, \tilde{\nu} \leq \nu$. Take a sequence $\tilde{\mu}^n, \tilde{\nu}^n$ such that $a^2 (|\mu - \tilde{\mu}^n| + |\nu - \tilde{\nu}^n|)^2 + b^2 W_2^2(\tilde{\mu}^n, \tilde{\nu}^n) \rightarrow T_2^{a,b}(\mu, \nu)$. Since $\tilde{\mu}^n \leq \mu$, then the sequence has uniformly bounded mass, bounded by $|\mu|$. Then there exists a subsequence weakly converging to some μ^* , i.e. $\tilde{\mu}^n \rightharpoonup \mu^*$, due to weak compactness (see [3]). The same holds for $\tilde{\nu}^n \rightharpoonup \nu^*$. One can then prove that

$$a^2 (|\mu - \tilde{\mu}^n| + |\nu - \tilde{\nu}^n|)^2 + b^2 W_2^2(\tilde{\mu}^n, \tilde{\nu}^n) \rightarrow a^2 (|\mu - \mu^*| + |\nu - \nu^*|)^2 + b^2 W_2^2(\mu^*, \nu^*).$$

Property 3. Symmetry is evident.

We now prove that $W_2^{a,b}(\mu, \nu) = 0$ implies $\mu = \nu$. Let $\tilde{\mu}, \tilde{\nu}$ realize the infimum in (4). We clearly have $|\mu - \tilde{\mu}| = |\nu - \tilde{\nu}| = 0$ that implies $\mu = \tilde{\mu}$ and $\nu = \tilde{\nu}$. Since W_2 is a distance, then $W_2(\tilde{\mu}, \tilde{\nu}) = 0$ implies $\tilde{\mu} = \tilde{\nu}$, that gives the result.

We now prove the triangular inequality $W_2^{a,b}(\mu, \eta) \leq W_2^{a,b}(\mu, \nu) + W_2^{a,b}(\nu, \eta)$. Let $(\tilde{\mu}, \tilde{\nu}_1)$ realizing $W_2^{a,b}(\mu, \nu)$ and $(\tilde{\nu}_2, \tilde{\eta})$ realizing $W_2^{a,b}(\nu, \eta)$. Also denote with π^1, π^2 the transference plans realizing $W_2(\tilde{\mu}, \tilde{\nu}_1), W_2(\tilde{\nu}_2, \tilde{\eta})$, respectively. Define now $\bar{\nu} := \min \{D_{\tilde{\nu}_2} \tilde{\nu}_1, 1\} \tilde{\nu}_2$. Define $\bar{\mu}, \bar{\eta}$ the marginals of $\bar{\nu}$ with respect to π^1, π^2 respectively, i.e. $\pi^1 \in \Pi(\bar{\mu}, \bar{\nu})$ and $\pi^2 \in \Pi(\bar{\nu}, \bar{\eta})$. Thus we have $\pi^2 \circ \pi^1 \in \Pi(\bar{\mu}, \bar{\eta})$. One can prove that

$$\begin{aligned} |\mu - \bar{\mu}| + |\eta - \bar{\eta}| &= |\mu - \tilde{\mu}| + |\tilde{\mu} - \bar{\mu}| + |\eta - \tilde{\eta}| + |\tilde{\eta} - \bar{\eta}| \leq |\mu - \tilde{\mu}| + |\nu - \tilde{\nu}_2| + |\eta - \tilde{\eta}| + |\nu - \tilde{\nu}_1| \\ W_2(\bar{\mu}, \bar{\eta}) &\leq W_2(\bar{\mu}, \bar{\nu}) + W_2(\bar{\nu}, \bar{\eta}) \leq W_2(\tilde{\mu}, \tilde{\nu}_1) + W_2(\tilde{\nu}_2, \tilde{\eta}). \end{aligned}$$

Rearranging terms, one has

$$\begin{aligned} T_2^{a,b}(\mu, \eta) &\leq a^2 (|\mu - \tilde{\mu}| + |\nu - \tilde{\nu}_1|)^2 + b^2 W_2^2(\tilde{\mu}, \tilde{\nu}_1) + a^2 (|\nu - \tilde{\nu}_2| + |\eta - \tilde{\eta}|)^2 + b^2 W_2^2(\tilde{\nu}_2, \tilde{\eta}) + \\ &\quad + 2a^2 (|\mu - \tilde{\mu}| + |\nu - \tilde{\nu}_1|) (|\nu - \tilde{\nu}_2| + |\eta - \tilde{\eta}|) + 2b^2 W_2(\tilde{\mu}, \tilde{\nu}_1) W_2(\tilde{\nu}_2, \tilde{\eta}). \end{aligned}$$

Apply at the last two terms of the right hand side the estimate $xy + zw \leq \sqrt{x^2 + z^2} \sqrt{y^2 + w^2}$ for positive numbers, that gives

$$T_2^{a,b}(\mu, \eta) \leq T_2^{a,b}(\mu, \nu) + T_2^{a,b}(\nu, \eta) + 2\sqrt{T_2^{a,b}(\mu, \nu)} \sqrt{T_2^{a,b}(\nu, \eta)} = \left(W_2^{a,b}(\mu, \nu) + W_2^{a,b}(\nu, \eta) \right)^2,$$

from which the triangular inequality easily follows.

Property 4. Take the choice $\tilde{\mu} = \tilde{\nu} = 0$ to estimate the infimum in $T_2^{a,b}$. □

2.3 Topology of the generalized Wasserstein distance

In this section we recall some useful topological results related to the metric space \mathcal{M} when endowed with the generalized Wasserstein distance.

We first define tightness in this context.

Definition 3 A set of measures M is tight if for each $\varepsilon > 0$ there exists a compact K_ε such that $\mu(\mathbb{R}^d \setminus K_\varepsilon) < \varepsilon$ for all $\mu \in M$.

We now recall the following important result about convergence with respect to the generalized Wasserstein distance, see [5, Theorem 13].

Theorem 4 Let $\{\mu_n\}$ be a sequence of measures in \mathbb{R}^d , and $\mu_n, \mu \in \mathcal{M}$. Then

$$W_2^{a,b}(\mu_n, \mu) \rightarrow 0 \quad \text{is equivalent to} \quad \mu_n \rightharpoonup \mu \quad \text{and} \quad \{\mu_n\} \text{ is tight.}$$

We finally recall the result of completeness, see [5, Proposition 15].

Proposition 5 The space \mathcal{M} endowed with the distance $W_2^{a,b}$ is a complete metric space.

2.4 Estimates of generalized Wasserstein distance under flow actions

In this section we give useful estimates for $W_2^{a,b}$ under flow actions. Observe that these properties are proved for measures $\mu, \nu \in \mathcal{M}_0^{ac}$ only. The main reason is that, in this setting, the standard Wasserstein distance is realized by a diffeomorphism, that is the solution of the Monge problem, see Section 2.1.

Proposition 6 Let v_t be a time-varying vector field, uniformly Lipschitz with respect to the space variable, and ϕ^t the flow generated by v . Let L be the Lipschitz constant of v , i.e. $|v_t(x) - v_t(y)| \leq L|x - y|$ for all t . Let $\mu, \nu \in \mathcal{M}_0^{ac}$. We have

- $W_2^{a,b}(\phi^t \# \mu, \phi^t \# \nu) \leq e^{\frac{3}{2}Lt} W_2^{a,b}(\mu, \nu)$
- $W_2^{a,b}(\mu, \phi^t \# \mu) \leq bt \|v\|_{C^0} |\mu|$

Proof. **Property 1.** Let $\tilde{\mu} \leq \mu, \tilde{\nu} \leq \nu$ realize $W_2^{a,b}(\mu, \nu)$, and T the map realizing $W_2(\tilde{\mu}, \tilde{\nu})$. Then we have

$$T_2^{a,b}(\phi^t \# \mu, \phi^t \# \nu) \leq a^2 (|\phi^t \# \mu - \phi^t \# \tilde{\mu}| + |\phi^t \# \nu - \phi^t \# \tilde{\nu}|)^2 + b^2 W_2^2(\phi^t \# \tilde{\mu}, \phi^t \# \tilde{\nu}).$$

Observe now that, since ϕ^t is a diffeomorphism and $\tilde{\mu} \leq \mu$, then $|\phi^t \# \mu - \phi^t \# \tilde{\mu}| = |\mu - \tilde{\mu}|$. Observe now that $W_2(\phi^t \# \tilde{\mu}, \phi^t \# \tilde{\nu}) \leq e^{\frac{3}{2}Lt} W_2(\tilde{\mu}, \tilde{\nu})$, as proved in [4, Prop. 1]. This gives the result.

Property 2. Choose $\tilde{\mu} = \mu$ and $\tilde{\nu} = \phi^t \# \mu$ to estimate $T_2^{a,b}(\mu, \phi^t \# \mu)$. This gives

$$T_2^{a,b}(\mu, \phi^t \# \mu) \leq b^2 W_2^2(\mu, \phi^t \# \mu) \leq b^2 |\mu| \int_{\mathbb{R}^d} |x - \phi^t(x)|^2 d\mu \leq b^2 |\mu| \int_{\mathbb{R}^d} (\|v\|_{C^0 t})^2 d\mu = b^2 |\mu|^2 \|v\|_{C^0}^2 t^2.$$

□

3 Generalized Benamou-Brenier formula

In this section we generalize the Benamou-Brenier formula (recalled below, see [2]) to $W_2^{a,b}$. We recall that the interest of such formula is to relate the Wasserstein distance between two measures μ_0, μ_1 to the minimization of the functional $\int |v_t|^2 d\mu_t$ among all solutions of the linear transport equation from μ_0 to μ_1 .

We first recall the original Benamou-Brenier formula. Observe that we deal with probability measures in \mathcal{M}_0^{ac} .

Theorem 7 *Let $\mu_0, \mu_1 \in \mathcal{P}_0^{ac}$ where $\mathcal{P}_0^{ac} := \mathcal{M}_0^{ac} \cap \mathcal{P}$ is the space of probability measures that are absolutely continuous with respect with the Lebesgue measure and with compact support. Endow \mathcal{P}_0^{ac} with the weak-* topology.*

Let $V(\mu_0, \mu_1)$ be the set of couples measure-velocity field $(\mu, v) := (\mu_t, v_t)_{t \in [0,1]}$ such that $\mu \in C([0,1], \mathcal{P}_0^{ac})$, $v \in L^2(d\mu_t dt)$, $\cup_{t \in [0,1]} \text{supp}(\mu_t)$ is bounded, and such that they satisfy the following boundary value problem

$$\begin{cases} \partial_t \mu_t + \nabla \cdot (v_t \mu_t) = 0 \\ \mu|_{t=0} = \mu_0, \quad \mu|_{t=1} = \mu_1. \end{cases}$$

Define the action functional $\mathcal{A}[\mu, v] := \int_0^1 dt \left(\int_{\mathbb{R}^d} d\mu_t |v_t|^2 \right)$ on $V(\mu_0, \mu_1)$. Then, it holds

$$W_2^2(\mu_0, \mu_1) = \inf \{ \mathcal{A}[\mu, v] \mid (\mu, v) \in V(\mu_0, \mu_1) \}. \quad (7)$$

Such result has been proven to hold also in the larger space of probability measures with finite second order moments, see [1]. It is also easy to prove that (7) holds for $\mu_0, \mu_1 \in \mathcal{M}_0^{ac}$ with the same mass m . Indeed, it is sufficient to use (7) for $m^{-1}\mu_0, m^{-1}\mu_1$ and to observe that we have the same degree of homogeneity on the left and right hand sides when multiplying by a constant.

We now prove that a similar result holds for $W_2^{a,b}$ and the transport equation with source. We first define the space and the functional that we study.

Definition 8 *Consider $\mu_0, \mu_1 \in \mathcal{M}_0^{ac}$. Let $V(\mu_0, \mu_1)$ be the set of triples (measure, velocity field, source term) $(\mu, v, h) := (\mu_t, v_t, h_t)_{t \in [0,1]}$ such that*

- $\mu \in C([0,1], \mathcal{M}_0^{ac})$, with \mathcal{M}_0^{ac} endowed with the weak-* topology;
- $v \in L^2(d\mu_t dt)$;
- $h \in L^1([0,1], \mathcal{M}_0^{ac})$ in the sense that $\int_0^1 dt \left(\int_{\mathbb{R}^d} d|h_t| \right) < \infty$;
- $\cup_{t \in [0,1]} \text{supp}(\mu_t)$ is bounded;

- they satisfy the following boundary value problem

$$\begin{cases} \partial_t \mu_t + \nabla \cdot (v_t \mu_t) = h_t, \\ \mu|_{t=0} = \mu_0, \quad \mu|_{t=1} = \mu_1. \end{cases} \quad (8)$$

We define the action functional

$$\mathcal{B}^{a,b}[\mu, v, h] := a^2 \left(\int_0^1 dt \left(\int_{\mathbb{R}^d} d|h_t| \right) \right)^2 + b^2 \int_0^1 dt \left(\int_{\mathbb{R}^d} d\mu_t |v_t|^2 \right).$$

Remark 9 Observe that the conditions given above also imply that $\cup_{t \in [0,1]} \text{supp}(h_t) \subset \cup_{t \in [0,1]} \text{supp}(\mu_t)$, and in particular h_t have uniformly bounded support. Indeed, by contradiction, assume that $\cup_{t \in [0,1]} \text{supp}(h_t) \not\subset \cup_{t \in [0,1]} \text{supp}(\mu_t)$. Looking at h as a functional on C_0^∞ functions, this means that there exists a function $\psi \in C_0^\infty([0,1] \times \mathbb{R}^d, \mathbb{R})$ with $\text{supp}(\psi) \subset [0,1] \times (\mathbb{R}^d \setminus (\cup_{t \in [0,1]} \text{supp}(\mu_t)))$ and such that $\int_0^1 dt \int_{\mathbb{R}^d} dh_t \psi(t, x) \neq 0$. Observe now that, by construction, one has $\int_0^1 dt \int d\mu_t (\partial_t \psi + v \cdot \nabla \psi) = 0$, since ψ and its derivatives are identically 0 on the support of μ_t for each $t \in [0,1]$. Observe now that (μ, v, h) satisfy (1) in the weak sense. Choosing ψ as a test function, one has $0 = \int_0^1 dt \int_{\mathbb{R}^d} dh_t \psi(t, x)$. Contradiction.

We now state the generalized Benamou-Brenier formula. We have

Theorem 10 Let $\mu_0, \mu_1 \in \mathcal{M}_0^{ac}$. Then

$$\inf \{ \mathcal{B}^{a,b}[\mu, v, h] \mid (\mu, v, h) \in V(\mu_0, \mu_1) \} = T_2^{a,b}(\mu_0, \mu_1). \quad (9)$$

It is clear the similarity between $\mathcal{B}^{a,b}$ and \mathcal{A} . In particular, the standard Benamou-Brenier formula can be recovered as a particular case of Theorem 10 when $h \equiv 0$ and $a \rightarrow \infty$.

To prove Theorem 10, we first need a series of lemmas, that we state and prove in the following section. We will then give the proof of Theorem 10 in Section 5.

4 Approximation of solutions of (1)

In this section, we give technical results that will be useful to prove Theorem 10. The idea is to approximate solutions of (1) via an adapted sample-and-hold method, and to prove the inequality $\mathcal{B}^{a,b}[\mu, v, h] \geq T_2^{a,b}(\mu_0, \mu_1)$ for such approximations.

We first estimate the difference $\int f d\mu - \int f d\nu$ in terms of $W_2^{a,b}(\mu, \nu)$.

Lemma 11 Let $\mu, \nu \in \mathcal{M}_0^{ac}$, and $f \in \text{Lip}(\mathbb{R}^d, \mathbb{R}) \cap L^\infty(\mathbb{R}^d, \mathbb{R})$. Then

$$\left| \int f d\mu - \int f d\nu \right| \leq \sqrt{2} \max \left\{ \frac{\|f\|_\infty}{a}, \frac{\|f\|_{Lip}}{b} \right\} W_2^{a,b}(\mu, \nu). \quad (10)$$

Proof. Let $\tilde{\mu} \leq \mu, \tilde{\nu} \leq \nu$ realizing $W_2^{a,b}(\mu, \nu)$. We have

$$\begin{aligned} \left| \int f d\mu - \int f d\nu \right| &\leq \left| \int f d(\mu - \tilde{\mu}) \right| + \left| \int f d(\tilde{\mu} - \tilde{\nu}) \right| + \left| \int f d(\tilde{\nu} - \nu) \right| \leq \\ &\leq \|f\|_\infty |\mu - \tilde{\mu}| + \|f\|_{Lip} W_1(\tilde{\mu}, \tilde{\nu}) + \|f\|_\infty |\tilde{\nu} - \nu|, \end{aligned} \quad (11)$$

where we have used that $|\mu| = \sup \{ \int f d\mu \mid \|f\|_\infty = 1 \}$ and the Kantorovich-Rubinstein duality formula $W_1(\mu, \nu) = \sup \{ \int f d(\mu - \nu) \mid \|f\|_{Lip} = 1 \}$. Recall that $W_1(\tilde{\mu}, \tilde{\nu}) \leq W_p(\tilde{\mu}, \tilde{\nu})$ for $p \geq 1$, see e.g. [7, Sec. 7.1.2]. Then (10) is a direct consequence of (11), by using $(x+y)^2 \leq 2(x^2 + y^2)$. \square

We now prove the inequality $\mathcal{B}^{a,b}[\mu, v, h] \geq T_2^{a,b}(\mu_0, \mu_1)$ under stronger regularity assumptions for v, h .

Lemma 12 Let $\mu_0, \mu_1 \in \mathcal{M}_0^{ac}$ and $(\mu, v, h) \in V(\mu_0, \mu_1)$ such that they satisfy assumptions of Definition 8 and additionally

- v is uniformly L -Lipschitz with respect to x ; it has C^0 -norm uniformly bounded in time, i.e. $M := \sup_{t \in [0,1]} \|v_t\|_{C^0} < \infty$;
- $h \in L^\infty([0,1], \mathcal{M}_0^{ac})$, i.e. it satisfies $P := \sup_{t \in [0,1]} \int_{\mathbb{R}^d} d|h_t(\cdot)| < \infty$.

Then

$$\mathcal{B}^{a,b}[\mu, v, h] \geq T_2^{a,b}(\mu_0, \mu_1). \quad (12)$$

Proof. The proof is divided into two main parts. In the first part we define $\mu^{[k]}$, an approximation of μ with a variation of the sample-and-hold method, and prove appropriate convergence results of $\mu^{[k]}$ to μ and of $\mathcal{B}^{a,b}[\mu^{[k]}, v^k, h^k]$ to $\mathcal{B}^{a,b}[\mu, v, h]$ for $k \rightarrow \infty$. In the second part, we define a new sequence $(\tilde{\mu}^{[k]}, \tilde{v}^k, \tilde{h}^k)$ such that $\mathcal{B}^{a,b}[\tilde{\mu}^{[k]}, \tilde{v}^k, \tilde{h}^k] \leq \mathcal{B}^{a,b}[\mu^{[k]}, v^k, h^k]$ and we prove that (12) holds for $\tilde{\mu}^{[k]}$. A simple inequality will then give the result for general μ .

Before the main parts of the proof, we state some simple remarks. First of all, since we deal with approximations of the dynamics given by v, h , then the approximated solution $\mu^{[k]}$ could fail to be a positive measure for some times. Then, one needs to replace $\mu^{[k]}$ with its positive part all along the proof. For simplicity of notation, this replacement is implicit all along the proof.

Second, we fix some notations that will be useful all along the proof. Given the initial datum μ_0 , we will prove that all measures studied in the proof have bounded mass, and in particular $|\mu_t|, |\mu_t^{[k]}|, |\tilde{\mu}_t^{[k]}| \leq |\mu_0| + P$. We define

$$m := |\mu_0| + P.$$

We also define

$$\alpha := \sqrt{2} \max \left\{ \frac{M}{a}, \frac{L}{b} \right\}, \quad \beta := 2aP + bMm.$$

Part 1: We divide this part of the proof in four parts. In the first, we define $\mu^{[k]}$, together with v^k, h^k . In the second, we prove that $\mu^{[k]}$ is a Cauchy sequence in a complete space, then it admits a limit μ^* . In the third, we prove that $\mu^* = \mu$, by proving that μ^* is the unique solution of (1). In the fourth, we prove that $\mathcal{B}^{a,b}[\mu^{[k]}, v^k, h^k]$ converges to $\mathcal{B}^{a,b}[\mu, v, h]$ for $k \rightarrow \infty$.

Part 1.1: We first define $\mu^{[k]}$, together with v^k, h^k . The main advantage of the definition given below is that v^k and h^k will never act at the same time, i.e. $v_t^k \neq 0$ implies $h^k = 0$ and viceversa. This helps both in the definition of $\mu^{[k]}$ and in the computation of $\mathcal{B}^{a,b}$. A scheme of the evolution of the mass $|\mu_t^{[k]}|$ is given in Figure 1.

Fix $k \in \mathbb{N}$ and define $\Delta t := 2^{-k}$. We discretize the time interval $[0, 1]$ in small intervals $[n\Delta t, (n+1)\Delta t]$. The idea of the discretization is first to divide each interval $[n\Delta t, (n+1)\Delta t]$ in three parts:

$$[n\Delta t, n\Delta t + \Delta t^2], \quad [n\Delta t + \Delta t^2, (n+1)\Delta t - \Delta t^2], \quad [(n+1)\Delta t - \Delta t^2, (n+1)\Delta t].$$

On the first part we use the negative part h^- of h , then the velocity v , then the positive part h^+ of h . Clearly, each term must be correctly rescaled, to have $\mu_{(n+1)\Delta t}^{[k]}$ close to $\mu_{(n+1)\Delta t}$.

We define the following vector field and the source term:

$$v_{n\Delta t+\tau}^k := \begin{cases} \frac{\Delta t}{\Delta t - 2\Delta t^2} v_{n\Delta t + \frac{\Delta t}{\Delta t - 2\Delta t^2}(\tau - \Delta t^2)} & \text{for } \tau \in (\Delta t^2, \Delta t - \Delta t^2], \\ 0 & \text{for } \tau \in (0, \Delta t^2] \cup (\Delta t - \Delta t^2, \Delta t], \end{cases}$$

$$h_{n\Delta t+\tau}^k := \begin{cases} \Delta t^{-1} h_{n\Delta t + \Delta t^{-1}\tau}^- & \text{for } \tau \in (0, \Delta t^2], \\ 0 & \text{for } \tau \in (\Delta t^2, \Delta t], \end{cases}$$

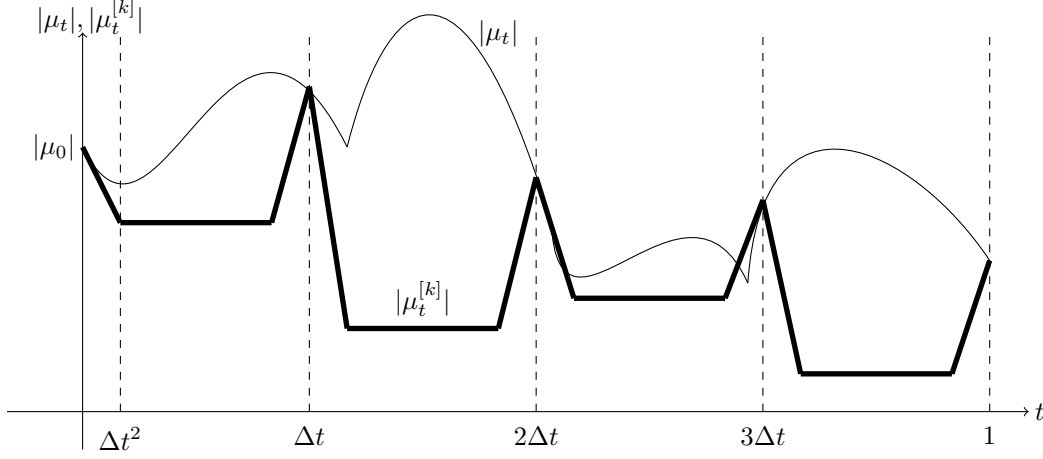


Figure 1: Evolution of $|\mu_t|, |\mu_t^{[k]}|$ for $k = 2$.

and

$$\bar{h}_{n\Delta t + \tau}^k := \begin{cases} 0 & \text{for } \tau \in (0, \Delta t - \Delta t^2], \\ \Delta t^{-1} h_{n\Delta t + \Delta t^{-1}(\tau - (\Delta t - \Delta t^2))}^+ & \text{for } \tau \in (\Delta t - \Delta t^2, \Delta t]. \end{cases}$$

We now introduce the following useful notations:

- $\Phi_{[t_1, t_2]}^k$ is the diffeomorphism corresponding to the flow generated by v^k on the time interval $[t_1, t_2]$;
- $\underline{H}_{[t_1, t_2]}^k := \int_{t_1}^{t_2} \underline{h}_t^k dt$ is the mass removal given by \underline{h}^k on the time interval $[t_1, t_2]$;
- $\bar{H}_{[t_1, t_2]}^k := \int_{t_1}^{t_2} \bar{h}_t^k dt$ is the mass creation given by \bar{h}^k on the time interval $[t_1, t_2]$.

Several properties of composition of these operators will be useful in the following. First, for n integer, one has:

- $\Phi_{[n\Delta t, (n+2)\Delta t]}^k = \Phi_{[(n+1)\Delta t, (n+2)\Delta t]}^k \circ \Phi_{[n\Delta t, (n+1)\Delta t]}^k$;
- $\underline{H}_{[n\Delta t, (n+2)\Delta t]}^k = \underline{H}_{[n\Delta t, (n+1)\Delta t]}^k + \underline{H}_{[(n+1)\Delta t, (n+2)\Delta t]}^k$, and similarly for $\bar{H}_{[n\Delta t, (n+2)\Delta t]}^k$.

Second, for n even, one also has, for $\Delta t = 2^{-k}$,

- $\Phi_{[n\Delta t, (n+2)\Delta t]}^{k-1} = \Phi_{[n\Delta t, (n+2)\Delta t]}^k$;
- $\underline{H}_{[n\Delta t, (n+2)\Delta t]}^{k-1} = \underline{H}_{[n\Delta t, (n+2)\Delta t]}^k$, and similarly for $\bar{H}_{[n\Delta t, (n+2)\Delta t]}^{k-1}$.

Proofs are direct consequences of the definitions and of standard properties of composition of flows.

We now define $\mu^{[k]}$. First define $\mu_0^{[k]} := \mu_0$. Then, for each interval $(n\Delta t, (n+1)\Delta t]$, define:

$$\mu_{n\Delta t + \tau}^{[k]} := \begin{cases} \mu_{n\Delta t}^{[k]} - \underline{H}_{[n\Delta t, n\Delta t + \tau]}^k & \text{for } \tau \in (0, \Delta t^2], \\ \Phi_{[n\Delta t, n\Delta t + \tau]}^k \# \mu_{n\Delta t}^{[k]} - \Phi_{[n\Delta t, n\Delta t + \tau]}^k \# \underline{H}_{[n\Delta t, (n+1)\Delta t]}^k & \text{for } \tau \in (\Delta t^2, \Delta t - \Delta t^2], \\ \mu_{n\Delta t + \Delta t - \Delta t^2}^{[k]} + \bar{H}_{[n\Delta t, n\Delta t + \tau]}^k & \text{for } \tau \in (\Delta t - \Delta t^2, \Delta t]. \end{cases}$$

It is easy to prove that $\mu^{[k]}$ is a solution of (1) in $C([0, 1], \mathcal{M}_0^{ac})$ with velocity field v^k , source $h^k := \bar{h}^k - \underline{h}^k$, and initial datum $\mu_0^{[k]} = \mu_0$. It is evident that the measure has uniformly bounded mass, in particular $|\mu_t^{[k]}| \leq m$ for all $t \in [0, 1]$.

It is also easy to prove the following property for $\mu^{[k]}$:

$$W_2^{a,b} \left(\mu_{n\Delta t}^{[k]}, \mu_{n\Delta t+\tau}^{[k]} \right) \leq \begin{cases} a|\underline{H}_{[n\Delta t, n\Delta t+\tau]}^k| \leq a\Delta t^{-1}\tau P & \text{for } \tau \in (0, \Delta t^2], \\ b\frac{(\tau-\Delta t^2)\Delta t}{\Delta t-2\Delta t^2}Mm + a\Delta tP & \text{for } \tau \in (\Delta t^2, \Delta t - \Delta t^2], \\ b\Delta tMm + a\Delta tP + a\Delta t^{-1}(\tau - \Delta t + \Delta t^2)P & \text{for } \tau \in (\Delta t - \Delta t^2, \Delta t]. \end{cases} \quad (13)$$

As a consequence, one has for $\tau \in [0, \Delta t]$ the following estimate:

$$W_2^{a,b} \left(\mu_{n\Delta t}^{[k]}, \mu_{n\Delta t+\tau}^{[k]} \right) \leq \Delta t(2aP + bMm) =: \beta\Delta t. \quad (14)$$

Part 1.2: We now prove that $\mu^{[k]}$ is a Cauchy sequence with respect to the distance d defined in (6).

First observe that, by substitution, the following formula holds for $\mu_{(n+2)\Delta t}^{[k]}$:

$$\begin{aligned} \mu_{(n+2)\Delta t}^{[k]} &= \Phi_{[(n+1)\Delta t, (n+2)\Delta t]}^k \# \mu_{(n+1)\Delta t}^{[k]} - \Phi_{[(n+1)\Delta t, (n+2)\Delta t]}^k \# \underline{H}_{[(n+1)\Delta t, (n+2)\Delta t]}^k + \overline{H}_{[(n+1)\Delta t, (n+2)\Delta t]}^k = \\ &= \Phi_{[(n+1)\Delta t, (n+2)\Delta t]}^k \# \left(\Phi_{[n\Delta t, (n+1)\Delta t]}^k \# \left(\mu_{n\Delta t}^{[k]} - \underline{H}_{[n\Delta t, (n+1)\Delta t]}^k \right) + \overline{H}_{[n\Delta t, (n+1)\Delta t]}^k \right) + \\ &\quad - \Phi_{[(n+1)\Delta t, (n+2)\Delta t]}^k \# \underline{H}_{[(n+1)\Delta t, (n+2)\Delta t]}^k + \overline{H}_{[(n+1)\Delta t, (n+2)\Delta t]}^k = \\ &= \Phi_{[n\Delta t, (n+2)\Delta t]}^k \# \mu_{n\Delta t}^{[k]} - \Phi_{[n\Delta t, (n+2)\Delta t]}^k \# \underline{H}_{[n\Delta t, (n+1)\Delta t]}^k + \Phi_{[(n+1)\Delta t, (n+2)\Delta t]}^k \# \overline{H}_{[n\Delta t, (n+1)\Delta t]}^k + \\ &\quad - \Phi_{[(n+1)\Delta t, (n+2)\Delta t]}^k \# \underline{H}_{[(n+1)\Delta t, (n+2)\Delta t]}^k + \overline{H}_{[(n+1)\Delta t, (n+2)\Delta t]}^k. \end{aligned}$$

We also decompose $\mu_{(n+2)\Delta t}^{[k-1]}$ by using properties of composition of $\Phi^k, \underline{H}^k, \overline{H}^k$. This gives:

$$\begin{aligned} \mu_{(n+2)\Delta t}^{[k-1]} &= \Phi_{[n\Delta t, (n+2)\Delta t]}^k \# \mu_{n\Delta t}^{[k-1]} - \Phi_{[n\Delta t, (n+2)\Delta t]}^k \# \underline{H}_{[n\Delta t, (n+1)\Delta t]}^k + \\ &\quad - \Phi_{[(n+1)\Delta t, (n+2)\Delta t]}^k \# \Phi_{[n\Delta t, (n+1)\Delta t]}^k \# \underline{H}_{[(n+1)\Delta t, (n+2)\Delta t]}^k + \overline{H}_{[n\Delta t, (n+1)\Delta t]}^k + \overline{H}_{[(n+1)\Delta t, (n+2)\Delta t]}^k \end{aligned}$$

We now estimate $W_2^{a,b} \left(\mu_{(n+2)\Delta t}^{[k-1]}, \mu_{(n+2)\Delta t}^{[k]} \right)$ with respect to $W_2^{a,b} \left(\mu_{n\Delta t}^{[k-1]}, \mu_{n\Delta t}^{[k]} \right)$, i.e. the value of $W_2^{a,b}$ at the right extreme of the interval of discretization for $k-1$ with respect to its value at the left extreme. We choose n even. We have:

$$\begin{aligned} W_2^{a,b} \left(\mu_{(n+2)\Delta t}^{[k-1]}, \mu_{(n+2)\Delta t}^{[k]} \right) &\leq W_2^{a,b} \left(\Phi_{[n\Delta t, (n+2)\Delta t]}^k \# \mu_{n\Delta t}^{[k-1]}, \Phi_{[n\Delta t, (n+2)\Delta t]}^k \# \mu_{n\Delta t}^{[k]} \right) + \\ &+ W_2^{a,b} \left(\Phi_{[n\Delta t, (n+2)\Delta t]}^k \# \underline{H}_{[n\Delta t, (n+1)\Delta t]}^k, \Phi_{[n\Delta t, (n+2)\Delta t]}^k \# \underline{H}_{[n\Delta t, (n+1)\Delta t]}^k \right) + \\ &+ W_2^{a,b} \left(\Phi_{[(n+1)\Delta t, (n+2)\Delta t]}^k \# \Phi_{[n\Delta t, (n+1)\Delta t]}^k \# \underline{H}_{[(n+1)\Delta t, (n+2)\Delta t]}^k, \Phi_{[(n+1)\Delta t, (n+2)\Delta t]}^k \# \underline{H}_{[(n+1)\Delta t, (n+2)\Delta t]}^k \right) + \\ &+ W_2^{a,b} \left(\overline{H}_{[n\Delta t, (n+1)\Delta t]}^k, \Phi_{[(n+1)\Delta t, (n+2)\Delta t]}^k \# \overline{H}_{[n\Delta t, (n+1)\Delta t]}^k \right) + W_2^{a,b} \left(\overline{H}_{[(n+1)\Delta t, (n+2)\Delta t]}^k, \overline{H}_{[(n+1)\Delta t, (n+2)\Delta t]}^k \right) \leq \\ &\leq e^{\frac{3}{2}L\Delta t} W_2^{a,b} \left(\mu_{n\Delta t}^{[k-1]}, \mu_{n\Delta t}^{[k]} \right) + 0 + e^{\frac{3}{2}L\Delta t} W_2^{a,b} \left(\Phi_{[n\Delta t, (n+1)\Delta t]}^k \# \underline{H}_{[(n+1)\Delta t, (n+2)\Delta t]}^k, \underline{H}_{[(n+1)\Delta t, (n+2)\Delta t]}^k \right) + \\ &+ b\Delta tM(\Delta tP) + 0 \leq e^{\frac{3}{2}L\Delta t} W_2^{a,b} \left(\mu_{n\Delta t}^{[k-1]}, \mu_{n\Delta t}^{[k]} \right) + be^{\frac{3}{2}L\Delta t} \Delta tM(\Delta tP) + b\Delta tM(\Delta tP). \end{aligned} \quad (15)$$

We apply the last inequality recursively. First recall that $W_2^{a,b} \left(\mu_0^{[k-1]}, \mu_0^{[k]} \right) = 0$ and that, for a sufficiently big k , it holds $e^{3/2L\Delta t} \leq 1 + 3L\Delta t$ and $3L\Delta t \leq 1$. This gives

$$\begin{aligned} W_2^{a,b} \left(\mu_{n\Delta t}^{[k-1]}, \mu_{n\Delta t}^{[k]} \right) &\leq bMP\Delta t^2(2 + 3L\Delta t) \frac{(1 + 3L\Delta t)^{n/2} - 1}{1 + 3L\Delta t - 1} \leq 3bMP\Delta t \frac{e^{\frac{3}{2}nL\Delta t} - 1}{3L} \leq \\ &\leq bMP2^{-k} \frac{e^{\frac{3}{2}L} - 1}{L}, \end{aligned}$$

where we have used that $n\Delta t \leq 1$. Observe that the estimate is independent of n . Applying it recursively, one has

$$W_2^{a,b} \left(\mu_{n\Delta t}^{[k]}, \mu_{n\Delta t}^{[k+l]} \right) \leq \frac{bMP(e^{\frac{3}{2}L} - 1)}{L} 2^{-(k+1)} \frac{1 - 2^{-l/2}}{1 - 2^{-1/2}}.$$

Finally, take any $t \in [0, 1]$: for each integer k , let n_k be the biggest even number such that $n_k 2^{-k} \leq t$. It clearly holds $|t - n_k 2^{-k}| < 2^{-k+1}$. One has

$$\begin{aligned} W_2^{a,b} \left(\mu_t^{[k]}, \mu_t^{[k+l]} \right) &\leq W_2^{a,b} \left(\mu_t^{[k]}, \mu_{n_k 2^{-k}}^{[k]} \right) + W_2^{a,b} \left(\mu_{n_k 2^{-k}}^{[k]}, \mu_{n_k 2^{-k}}^{[k+l]} \right) + W_2^{a,b} \left(\mu_{n_k 2^{-k}}^{[k+l]}, \mu_t^{[k+l]} \right) \leq \\ &\leq 2\beta 2^{-k} + \frac{bMP(e^{\frac{3}{2}L} - 1)}{L} 2^{-(k+1)} \frac{1 - 2^{-l/2}}{1 - 2^{-1/2}} + 2\beta 2^{-k}, \end{aligned}$$

where we have used (14) twice for the first term and 2^{l+1} times for third term. Since the estimate does not depend on t , one has $d(\mu^{[k]}, \mu^{[k+l]}) \leq C_1 2^{-k}$ with $C_1 := 4\beta + \frac{bMP(e^{\frac{3}{2}L} - 1)}{L} 2^{-1} \frac{1}{1 - 2^{-1/2}}$. Since the estimate does not depend on l and $d(\mu^{[k]}, \mu^{[k+l]}) \rightarrow 0$ for $k \rightarrow \infty$, we have that $\mu^{[k]}$ is a Cauchy sequence. Since $C([0, 1], \mathcal{M})$ is complete with respect to d , then there exists a limit $\mu^* := \lim_{k \rightarrow \infty} \mu^{[k]}$, with $\mu^* \in \mathcal{M}$.

Part 1.3: We now prove that $\mu^* = \mu$. We prove it by proving that it is a weak solution of (1). By uniqueness the result will follow. We have to prove that, for any² $f_t \in C_0^\infty([0, 1] \times \mathbb{R}^d)$, it holds

$$\int_0^1 dt \left(\int_{\mathbb{R}^d} (d\mu_t^* (\partial_t f_t + v_t \cdot \nabla f_t) + dh_t f_t) \right) = 0. \quad (16)$$

Observe that $\mu^{[k]}$ is a solution of (1) with vector field v^k , and source $h^k := \bar{h}^k - \underline{h}^k$. Then

$$\int_0^1 dt \left(\int_{\mathbb{R}^d} (d\mu_t^{[k]} (\partial_t f_t + v_t^k \cdot \nabla f_t) + dh_t^k f_t) \right) = 0.$$

We then prove (16) by proving the three following limits:

$$\lim_k \left| \int_0^1 dt \left(\int_{\mathbb{R}^d} (d\mu_t^* - d\mu_t^{[k]}) \partial_t f_t \right) \right| = 0, \quad \lim_k \left| \int_0^1 dt \left(\int_{\mathbb{R}^d} d\mu_t^* v_t \cdot \nabla f_t - d\mu_t^{[k]} v_t^k \cdot \nabla f_t \right) \right| = 0,$$

$$\lim_k \left| \int_0^1 dt \left(\int_{\mathbb{R}^d} d(h_t - h_t^k) f_t \right) \right| = 0.$$

1. $\lim_k \left| \int_0^1 dt \left(\int_{\mathbb{R}^d} (d\mu_t^* - d\mu_t^{[k]}) \partial_t f_t \right) \right| = 0$. This is a consequence of (10). Indeed, one has

$$\begin{aligned} \lim_k \left| \int_0^1 dt \left(\int_{\mathbb{R}^d} (d\mu_t^* - d\mu_t^{[k]}) \partial_t f_t \right) \right| &\leq \int_0^1 dt \left(W_2^{a,b} \left(\mu_t^*, \mu_t^{[k]} \right) \sqrt{2} \max \left\{ \frac{\|\partial_t f_t\|_\infty}{a}, \frac{\|\partial_t f_t\|_{Lip}}{b} \right\} \right) \leq \\ &\leq d(\mu^*, \mu^{[k]}) \sqrt{2} \max \left\{ \frac{\|\partial_t f_t\|_\infty}{a}, \frac{\|\partial_t f_t\|_{Lip}}{b} \right\} \rightarrow 0 \end{aligned}$$

2. $\lim_k \left| \int_0^1 dt \left(\int_{\mathbb{R}^d} d\mu_t^* v_t \cdot \nabla f_t - d\mu_t^{[k]} v_t^k \cdot \nabla f_t \right) \right| = 0$. We first fix k and $\Delta t := 2^{-k}$, and estimate

$$\int_0^{\Delta t} dt \left(\int_{\mathbb{R}^d} d\mu_{n\Delta t+t}^* v_{n\Delta t+t} \cdot \nabla f_{n\Delta t+t} \right) - \int_0^{\Delta t} d\tau \left(\int_{\mathbb{R}^d} d\mu_{n\Delta t+\tau}^{[k]} v_{n\Delta t+\tau}^k \cdot \nabla f_{n\Delta t+\tau} \right). \quad (17)$$

²The index t will be useful in the following change of variable in time.

Using the definition of $v_{n\Delta t+\tau}^k$, we have that it is 0 for $\tau \in [0, \Delta t^2] \cup (\Delta t - \Delta t^2, \Delta t]$ and that for $\tau \in (\Delta t^2, \Delta t - \Delta t^2]$ it holds $v_{n\Delta t+\tau}^k = \frac{\Delta t}{\Delta t - 2\Delta t^2} v_{n\Delta t + \frac{\Delta t}{\Delta t - 2\Delta t^2}(\tau - \Delta t^2)}$. Then, after the change of variable $\tau \rightarrow t := (\tau - \Delta t^2) \frac{\Delta t}{\Delta t - 2\Delta t^2}$, we have

$$\begin{aligned} & \int_0^{\Delta t} d\tau \left(\int_{\mathbb{R}^d} d\mu_{n\Delta t+\tau}^{[k]} v_{n\Delta t+\tau}^k \cdot \nabla f_{n\Delta t+\tau} \right) = \\ &= \frac{\Delta t}{\Delta t - 2\Delta t^2} \int_{\Delta t^2}^{\Delta t - \Delta t^2} d\tau \left(\int_{\mathbb{R}^d} d\mu_{n\Delta t+\tau}^{[k]} v_{n\Delta t + \frac{\Delta t}{\Delta t - 2\Delta t^2}(\tau - \Delta t^2)} \cdot \nabla f_{n\Delta t+\tau} \right) = \\ &= \frac{\Delta t}{\Delta t - 2\Delta t^2} \int_0^{\Delta t} \frac{\Delta t - 2\Delta t^2}{\Delta t} dt \left(\int_{\mathbb{R}^d} d\mu_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t}^{[k]} v_{n\Delta t+t} \cdot \nabla f_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t} \right). \end{aligned}$$

To go back to (17), we estimate for each $t \in [0, \Delta t]$ the following quantity³:

$$\begin{aligned} & \left| \int_{\mathbb{R}^d} d\mu_{n\Delta t+t}^* v_{n\Delta t+t} \cdot \nabla f_{n\Delta t+t} - \int_{\mathbb{R}^d} d\mu_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t}^{[k]} v_{n\Delta t+t} \cdot \nabla f_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t} \right| \leq \\ & \left| \int_{\mathbb{R}^d} d\mu_{n\Delta t+t}^* v_{n\Delta t+t} \cdot \nabla f_{n\Delta t+t} - \int_{\mathbb{R}^d} d\mu_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t}^{[k]} v_{n\Delta t+t} \cdot \nabla f_{n\Delta t+t} \right| + \\ & + \left| \int_{\mathbb{R}^d} d\mu_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t}^{[k]} v_{n\Delta t+t} \cdot \nabla f_{n\Delta t+t} - \int_{\mathbb{R}^d} d\mu_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t}^{[k]} v_{n\Delta t+t} \cdot \nabla f_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t} \right| \leq \\ & W_2^{a,b} \left(\mu_{n\Delta t+t}^*, \mu_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t}^{[k]} \right) \sqrt{2} \max \left\{ \frac{M \|\nabla f_t\|_\infty}{a}, \frac{L \|\nabla f_t\|_{Lip}}{b} \right\} + \\ & + \left| \mu_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t}^{[k]} \right| M \|\nabla f_{n\Delta t+t} - \nabla f_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t}\|_\infty. \end{aligned} \quad (18)$$

We estimate the first term of the right hand side of (18) via

$$W_2^{a,b} \left(\mu_{n\Delta t+t}^*, \mu_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t}^{[k]} \right) \leq W_2^{a,b} \left(\mu_{n\Delta t+t}^*, \mu_{n\Delta t+t}^{[k]} \right) + W_2^{a,b} \left(\mu_{n\Delta t+t}^{[k]}, \mu_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t}^{[k]} \right).$$

We estimate $W_2^{a,b} \left(\mu_{n\Delta t+t}^{[k]}, \mu_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t}^{[k]} \right)$ by studying three cases:

- (a) $t \in [0, \Delta t^2]$: We observe that the evolution from $\mu_{n\Delta t+t}^{[k]}$ to $\mu_{n\Delta t+\Delta t^2}^{[k]}$ is given by removal of mass $\underline{H}_{[n\Delta t+t, n\Delta t+\Delta t^2]}^k$, while the evolution from $\mu_{n\Delta t+\Delta t^2}^{[k]}$ to $\mu_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t}^{[k]}$ is given by the push-forward of the diffeomorphism $\Phi_{[n\Delta t+\Delta t^2, n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t]}^k$. We then have

$$\begin{aligned} & W_2^{a,b} \left(\mu_{n\Delta t+t}^{[k]}, \mu_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t}^{[k]} \right) \leq W_2^{a,b} \left(\mu_{n\Delta t+t}^{[k]}, \mu_{n\Delta t+\Delta t^2}^{[k]} \right) + \\ & + W_2^{a,b} \left(\mu_{n\Delta t+\Delta t^2}^{[k]}, \mu_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t}^{[k]} \right) \leq |t - \Delta t^2| \Delta t^{-1} P + b \frac{\Delta t - 2\Delta t^2}{\Delta t} t \|v^k\|_{C^0 m} = \\ & = |t - \Delta t^2| \Delta t^{-1} P + b M m t. \end{aligned} \quad (19)$$

- (b) $t \in (\Delta t^2, \Delta t - \Delta t^2]$: We observe that the evolution is given by the push-forward of the diffeomorphism $\Phi_{[n\Delta t+t, n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t]}^k$. We have

$$\begin{aligned} & W_2^{a,b} \left(\mu_{n\Delta t+t}^{[k]}, \mu_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t}^{[k]} \right) \leq b \left| t - \left(\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t} t \right) \right| \|v^k\|_{C^0 m} \leq \\ & \leq b |2t\Delta t - \Delta t^2| \frac{\Delta t}{\Delta t - 2\Delta t^2} M m. \end{aligned}$$

³Here we denote with $\|\nabla f_t\|_{Lip}$ the Lipschitz constant for ∇f_t with respect to all t, x -variables, even if for (18) the Lipschitz constant in space is needed only.

(c) $t \in [\Delta t - \Delta t^2, \Delta t]$: This is similar to case 1. We have

$$\begin{aligned} W_2^{a,b} \left(\mu_{n\Delta t+t}^{[k]}, \mu_{n\Delta t+\Delta t^2+\frac{\Delta t-2\Delta t^2}{\Delta t}t}^{[k]} \right) &\leq |t - \Delta t + \Delta t^2| \Delta t^{-1} P + \\ &+ b \left| \frac{\Delta t - 2\Delta t^2}{\Delta t} t - (\Delta t - 2\Delta t^2) \right| Mm. \end{aligned} \quad (20)$$

We estimate the second term of the right hand side of (18) via⁴ $\left| \mu_{n\Delta t+\Delta t^2+\frac{\Delta t-2\Delta t^2}{\Delta t}t}^{[k]} \right| \leq m$ and

$$\|\nabla f_{n\Delta t+t} - \nabla f_{n\Delta t+\Delta t^2+\frac{\Delta t-2\Delta t^2}{\Delta t}t}\|_\infty \leq \|\nabla f_t\|_{Lip} \left| t - \left(\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t} t \right) \right| = \|\nabla f_t\|_{Lip} |2t\Delta t - \Delta t^2|.$$

Observe that both terms of the right hand side of (18) have a symmetry property: the value in t coincides with the value in $\Delta t - t$.

We go back to (17) and, by using (18) and the symmetry described above, we have

$$\begin{aligned} &\left| \int_0^{\Delta t} dt \left(\int_{\mathbb{R}^d} d\mu_{n\Delta t+t}^* v_{n\Delta t+t} \cdot \nabla f_{n\Delta t+t} \right) - \int_0^{\Delta t} d\tau \left(\int_{\mathbb{R}^d} d\mu_{n\Delta t+\tau}^{[k]} v_{n\Delta t+\tau}^k \cdot \nabla f_{n\Delta t+\tau} \right) \right| \leq \\ &\leq 2\sqrt{2} \max \left\{ \frac{M\|\nabla f_t\|_\infty}{a}, \frac{L\|\nabla f_t\|_{Lip}}{b} \right\} \int_0^{\Delta t} dt W_2^{a,b} \left(\mu_{n\Delta t+t}^*, \mu_{n\Delta t+t}^{[k]} \right) + \\ &+ 2\sqrt{2} \max \left\{ \frac{M\|\nabla f_t\|_\infty}{a}, \frac{L\|\nabla f_t\|_{Lip}}{b} \right\} \int_0^{\Delta t^2} dt (|t - \Delta t^2| \Delta t^{-1} P + bMmt) + \\ &+ 2\sqrt{2} \max \left\{ \frac{M\|\nabla f_t\|_\infty}{a}, \frac{L\|\nabla f_t\|_{Lip}}{b} \right\} \int_{\Delta t^2}^{\Delta t/2} dt b|2t\Delta t - \Delta t^2| \frac{\Delta t}{\Delta t - 2\Delta t^2} Mm + \\ &+ 2 \int_0^{\Delta t/2} dt mM \|\nabla f_t\|_{Lip} |2t\Delta t - \Delta t^2| \leq Cd(\mu^*, \mu^{[k]}) \Delta t + C\Delta t^3/2 + \\ &+ C(\Delta t - 2\Delta t^2) \Delta t^3/2 + C \int_0^{\Delta t/2} dt |2t\Delta t - \Delta t^2| + C \int_0^{\Delta t/2} dt |2t\Delta t - \Delta t^2| \end{aligned} \quad (21)$$

with $C = 2\sqrt{2} \max \left\{ \frac{M\|\nabla f_t\|_\infty}{a}, \frac{L\|\nabla f_t\|_{Lip}}{b}, \|\nabla f_t\|_{Lip} \right\} \cdot \max \{1, P, 2bMm, Mm\}$. The estimate holds for $k \geq 2$, for which it holds $\frac{\Delta t}{\Delta t - 2\Delta t^2} \leq 2$. We simply estimate (21) with $Cd(\mu^*, \mu^{[k]}) \Delta t + C\Delta t^3/2 + C\Delta t^4/2 + C\Delta t^3/2 + C\Delta t^3/2 < Cd(\mu^*, \mu^{[k]}) \Delta t + 3C\Delta t^3$, by using $|2t\Delta t - \Delta t^2| \leq \Delta t^2$.

Going back to our estimate, using (17) on each interval $[n\Delta t, (n+1)\Delta t]$, we have

$$\begin{aligned} \lim_k \left| \int_0^1 dt \left(\int_{\mathbb{R}^d} d\mu_t^* v_t \cdot \nabla f_t - d\mu_t^{[k]} v_t^k \cdot \nabla f_t \right) \right| &\leq \lim_k \sum_{n=0}^{2^k-1} \left| \int_0^{\Delta t} dt \left(\int_{\mathbb{R}^d} d\mu_{n\Delta t+t}^* v_{n\Delta t+t} \cdot \nabla f_{n\Delta t+t} \right) + \right. \\ &\left. - \int_0^{\Delta t} d\tau \left(\int_{\mathbb{R}^d} d\mu_{n\Delta t+\tau}^{[k]} v_{n\Delta t+\tau}^k \cdot \nabla f_{n\Delta t+\tau} \right) \right| \leq \lim_k 2^k (Cd(\mu^*, \mu^{[k]}) 2^{-k} + 3C2^{-3k}) = \lim_k Cd(\mu^*, \mu^{[k]}) = 0. \end{aligned}$$

3. $\lim_k \left| \int_0^1 dt \left(\int_{\mathbb{R}^d} d(h_t - h_t^k) f_t \right) \right| = 0$. We first fix k and $\Delta t := 2^{-k}$, and estimate the negative part

$$\left| \int_0^{\Delta t} dt \left(\int_{\mathbb{R}^d} dh_{n\Delta t+t}^- f_{n\Delta t+t} \right) - \int_0^{\Delta t} d\tau \left(\int_{\mathbb{R}^d} dh_{n\Delta t+\tau}^k f_{n\Delta t+\tau} \right) \right|. \quad (22)$$

⁴Here it is sufficient to use the Lipschitz constant in the time variable.

By using the definition of \underline{h}^k and the change of variable $t = \Delta t^{-1}\tau$, we have

$$\int_0^{\Delta t} d\tau \left(\int_{\mathbb{R}^d} d\underline{h}_{n\Delta t+\tau}^k f_{n\Delta t+\tau} \right) = \Delta t^{-1} \int_0^{\Delta t^2} d\tau \left(\int_{\mathbb{R}^d} d\underline{h}_{n\Delta t+\Delta t^{-1}\tau}^- f_{n\Delta t+\tau} \right) = \int_0^{\Delta t} dt \left(\int_{\mathbb{R}^d} d\underline{h}_{n\Delta t+t}^- f_{n\Delta t+t\Delta t} \right).$$

Going back to (22), we estimate it by

$$\left| \int_0^{\Delta t} dt \left(\int_{\mathbb{R}^d} d\underline{h}_{n\Delta t+t}^- (f_{n\Delta t+t} - f_{n\Delta t+t\Delta t}) \right) \right| \leq \int_0^{\Delta t} dt P \|f_t\|_{Lip}(t - t\Delta t) = P \|f_t\|_{Lip}(1 - \Delta t) \frac{\Delta t^2}{2}.$$

The same estimate can be proved for the positive part. Going back to our estimate, we have

$$\begin{aligned} \lim_k \left| \int_0^1 dt \left(\int_{\mathbb{R}^d} d(h_t - h_t^k) f_t \right) \right| &\leq \lim_k \sum_{n=0}^{2^k-1} \left| \int_0^{\Delta t} dt \left(\int_{\mathbb{R}^d} dh_{n\Delta t+t} f_{n\Delta t+t} \right) - \int_0^{\Delta t} d\tau \left(\int_{\mathbb{R}^d} dh_{n\Delta t+\tau}^k f_{n\Delta t+\tau} \right) \right| \leq \\ &\leq \lim_k \sum_{n=0}^{2^k-1} \left| \int_0^{\Delta t} dt \left(\int_{\mathbb{R}^d} dh_{n\Delta t+t}^+ f_{n\Delta t+t} \right) - \int_0^{\Delta t} d\tau \left(\int_{\mathbb{R}^d} d\bar{h}_{n\Delta t+\tau}^k f_{n\Delta t+\tau} \right) \right| + \\ &+ \lim_k \sum_{n=0}^{2^k-1} \left| \int_0^{\Delta t} dt \left(\int_{\mathbb{R}^d} d\underline{h}_{n\Delta t+t}^- f_{n\Delta t+t} \right) - \int_0^{\Delta t} d\tau \left(\int_{\mathbb{R}^d} d\underline{h}_{n\Delta t+\tau}^k f_{n\Delta t+\tau} \right) \right| \leq \\ &\leq \lim_k 2 \cdot 2^k P \|f_t\|_{Lip}(1 - \Delta t) \frac{2^{-2k}}{2} = 0 \end{aligned}$$

We have proved that μ^* is a solution of (1), with $\mu^* \in C([0, 1], \mathcal{M})$. Observe now that $\mu^* - \mu$ is a solution of (1) with initial datum 0, vector field v_t and source 0. Applying standard result of existence and uniqueness of solutions of (1) with zero source in $C([0, 1], \mathcal{M})$, we have $\mu_t^* = \mu$. Since $\mu \in C([0, 1], \mathcal{M}_0^{ac})$, then $\mu^* \in C([0, 1], \mathcal{M}_0^{ac})$ too.

Part 1.4: We now prove that $\mathcal{B}^{a,b}[\mu^{[k]}, v^k, h^k] \rightarrow \mathcal{B}^{a,b}[\mu, v, h]$ for $k \rightarrow \infty$. This will be a consequence of the two following results:

- We first estimate

$$\begin{aligned} &\left| \int_0^{\Delta t} dt \left(\int_{\mathbb{R}^d} d\mu_{n\Delta t+t} |v_{n\Delta t+t}|^2 \right) - \int_0^{\Delta t} d\tau \left(\int_{\mathbb{R}^d} d\mu_{n\Delta t+\tau}^{[k]} |v_{n\Delta t+\tau}^k|^2 \right) \right| \leq \\ &\left| \int_0^{\Delta t} dt \left(\int_{\mathbb{R}^d} d\mu_{n\Delta t+t} |v_{n\Delta t+t}|^2 \right) - \int_0^{\Delta t} dt \left(\int_{\mathbb{R}^d} d\mu_{n\Delta t+t}^{[k]} |v_{n\Delta t+t}|^2 \right) \right| + \\ &+ \left| \int_0^{\Delta t} dt \left(\int_{\mathbb{R}^d} d\mu_{n\Delta t+t}^{[k]} |v_{n\Delta t+t}|^2 \right) - \int_0^{\Delta t} d\tau \left(\int_{\mathbb{R}^d} d\mu_{n\Delta t+\tau}^{[k]} |v_{n\Delta t+\tau}^k|^2 \right) \right| \end{aligned} \quad (23)$$

For the first term, observe that $|v_{n\Delta t+t}|^2$ is bounded with bound M^2 and Lipschitz in space with Lipschitz constant $2LM$. Then we can estimate the first term with

$$\int_0^{\Delta t} dt \sqrt{2} \max \left\{ \frac{M^2}{a}, \frac{2LM}{b} \right\} W_2^{a,b}(\mu_t, \mu_t^{[k]}) \leq 2M\alpha(\mu, \mu^{[k]})\Delta t.$$

For the second term, we replace $v_{n\Delta t+\tau}^k$ with its definition and apply the change of variable $\tau \rightarrow t = (\tau - \Delta t^2) \frac{\Delta t}{\Delta t - 2\Delta t^2}$. This gives

$$\begin{aligned} &\left| \int_0^{\Delta t} dt \left(\int_{\mathbb{R}^d} d\mu_{n\Delta t+t}^{[k]} |v_{n\Delta t+t}|^2 \right) - \int_0^{\Delta t} dt \left(\int_{\mathbb{R}^d} d\mu_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t} t}^{[k]} |v_{n\Delta t+t}|^2 \right) \right| \leq \\ &\leq 2M\alpha \int_0^{\Delta t} dt W_2^{a,b} \left(\mu_{n\Delta t+t}^{[k]}, \mu_{n\Delta t+\Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t} t}^{[k]} \right). \end{aligned}$$

Observe that both $n\Delta t + t, n\Delta t + \Delta t^2 + \frac{\Delta t - 2\Delta t^2}{\Delta t}t \in [n\Delta t, (n+1)\Delta t]$. Then, using (14), we have

$$W_2^{a,b} \left(\mu_{n\Delta t+t}^{[k]}, \mu_{n\Delta t+\Delta t^2+\frac{\Delta t-2\Delta t^2}{\Delta t}t}^{[k]} \right) \leq 2\beta\Delta t.$$

Going back to (23), we estimate it with

$$2M\alpha d(\mu, \mu^{[k]})\Delta t + 4M\alpha\beta\Delta t^2 \leq Kd(\mu, \mu^{[k]})\Delta t + K\Delta t^2,$$

defining $K := \max\{2M\alpha, 4M\alpha\beta\}$.

- We now estimate

$$\left(\int_0^1 dt \left(\int_{\mathbb{R}^d} d|h_t| \right) \right)^2 - \left(\int_0^1 dt \left(\int_{\mathbb{R}^d} d|h_t^k| \right) \right)^2 = \left(\int_0^1 dt \left(\int_{\mathbb{R}^d} d|h_t| - d|h_t^k| \right) \right) \left(\int_0^1 dt \left(\int_{\mathbb{R}^d} d|h_t| + d|h_t^k| \right) \right).$$

We observe that, by definition of \underline{h}_t^k , one has $\int_{n\Delta t}^{(n+1)\Delta t} dt \int_{\mathbb{R}^d} d\bar{h}_t^- = \int_{n\Delta t}^{(n+1)\Delta t} dt \int_{\mathbb{R}^d} d\underline{h}_t^k$. The same result can be proved for the positive part. Applying it on each interval $[n\Delta t, (n+1)\Delta t]$, we have

$$\left(\int_0^1 dt \left(\int_{\mathbb{R}^d} d|h_t| \right) \right)^2 - \left(\int_0^1 dt \left(\int_{\mathbb{R}^d} d|h_t^k| \right) \right)^2 = 0.$$

We now prove that $\lim_k \mathcal{B}^{a,b}[\mu^{[k]}, v^k, h^k] = \mathcal{B}^{a,b}[\mu, v, h]$. Using the two previous estimates, we have

$$\left| \mathcal{B}^{a,b}[\mu^{[k]}, v^k, h^k] - \mathcal{B}^{a,b}[\mu, v, h] \right| \leq b^2 \sum_{n=0}^{2^k-1} K \left(\Delta t d(\mu, \mu^{[k]}) + \Delta t^2 \right) + 0 \cdot a^2 = b^2 K \left(d(\mu, \mu^{[k]}) + 2^{-k} \right).$$

Since $\lim_k d(\mu, \mu^{[k]}) = \lim_k d(\mu^*, \mu^{[k]}) = 0$, then $\lim_k \mathcal{B}^{a,b}[\mu^{[k]}, v^k, h^k] = \mathcal{B}^{a,b}[\mu, v, h]$.

Part 2: We now define a $\tilde{\mu}^{[k]}$, together with \tilde{v}^k, \tilde{h}^k , that satisfies the three following properties:

1. $\tilde{\mu}^{[k]}$ drives μ_0 to $\mu_1^{[k]}$, i.e. $(\tilde{\mu}^{[k]}, \tilde{v}^k, \tilde{h}^k) \in V(\mu_0, \mu_1^{[k]})$;
2. it holds $\mathcal{B}^{a,b}[\tilde{\mu}^{[k]}, \tilde{v}^k, \tilde{h}^k] \leq \mathcal{B}^{a,b}[\mu^{[k]}, v^k, h^k]$;
3. it holds $T_2^{a,b}(\mu_0, \mu_1^{[k]}) \leq \mathcal{B}^{a,b}[\tilde{\mu}^{[k]}, \tilde{v}^k, \tilde{h}^k]$.

We divide this part in three parts. In the first, we define $\tilde{\mu}^{[k]}$. In the second, we prove the properties stated above. In the third, we prove the final result (12).

Part 2.1: We now define $\tilde{\mu}^{[k]}$. We first recall that $(\mu^{[k]}, v^k, h^k)$ has a particular property: for each time t^* it exists at most one quantity $v_{t^*}^k, \underline{h}_{t^*}^k, \bar{h}_{t^*}^k$ that is nonzero. Moreover, if $v_{t^*}^k$ is nonzero, then there exists a whole interval $[t_1, t_1 + \Delta t - \Delta t^2]$ containing t^* such that $\underline{h}_t^k = 0, \bar{h}_t^k = 0$ for all $t \in [t_1, t_1 + \Delta t - \Delta t^2]$. Similar properties hold for $\underline{h}^k, \bar{h}^k$, with interval of length Δt^2 . We call such property **piecewise v-h action property**, PVHA for short.

We now define three transformations that preserve PVHA property. The transformation induced on the mass is described in Figure 2.

Transformation DOWN \mathcal{D} : Let (μ, v, h) be PVHA. Let \bar{t} be a time such that: $v_t = 0$ on the interval $[\bar{t} - \Delta t^2, \bar{t} + \Delta t^2]$; $\bar{h}_t^- = 0$ on the interval $[\bar{t} - \Delta t^2, \bar{t}]$; $\bar{h}_t^+ = 0$ on the interval $[\bar{t}, \bar{t} + \Delta t^2]$. Then replace h with \hat{h} defined as follows:

$$\hat{h}_t := \begin{cases} h_t & \text{for } t \in [0, \bar{t} - \Delta t^2] \cup (\bar{t} + \Delta t^2, 1], \\ h_{t+\Delta t^2} & \text{for } t \in (\bar{t} - \Delta t^2, \bar{t}], \\ h_{t-\Delta t^2} & \text{for } t \in (\bar{t}, \bar{t} + \Delta t^2]. \end{cases}$$

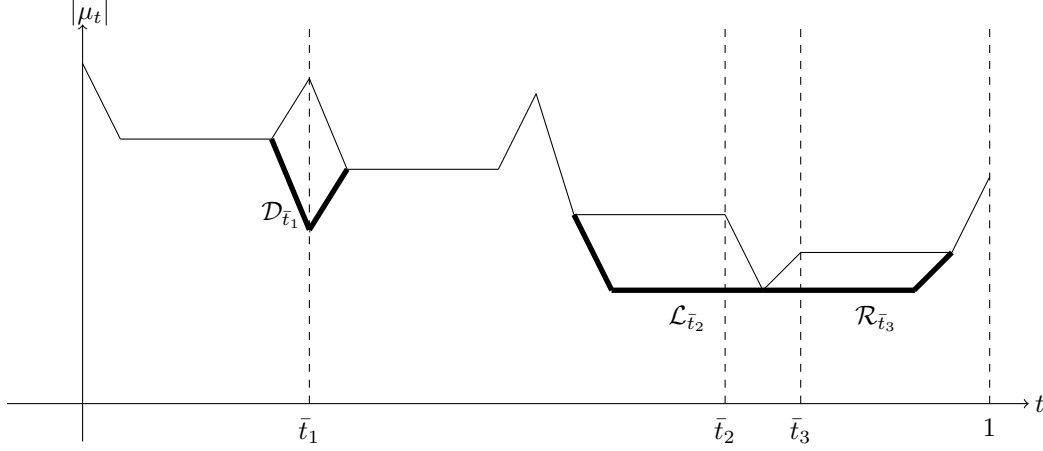


Figure 2: Transformations DOWN $\mathcal{D}_{\bar{t}_1}$, LEFT $\mathcal{L}_{\bar{t}_2}$ and RIGHT $\mathcal{R}_{\bar{t}_3}$.

Keep v . Then the solution $\hat{\mu}$ of (1) with velocity v and source \hat{h} satisfies

$$\hat{\mu}_t := \begin{cases} \mu_t & \text{for } t \in [0, \bar{t} - \Delta t^2] \cup (\bar{t} + \Delta t^2, 1], \\ \mu_{\bar{t} - \Delta t^2} - \int_{\bar{t}}^{t + \Delta t^2} ds h_s^- & \text{for } t \in (\bar{t} - \Delta t^2, \bar{t}], \\ \hat{\mu}_{\bar{t}} + \int_{\bar{t} - \Delta t^2}^{t - \Delta t^2} ds h_s^+ & \text{for } t \in (\bar{t}, \bar{t} + \Delta t^2]. \end{cases}$$

We use the notation $\mathcal{D}_{\bar{t}}$ for this transformation applied at a certain time \bar{t} , i.e. $\mathcal{D}_{\bar{t}}(\mu) := \hat{\mu}$ for the transformation of the measure and $\mathcal{D}_{\bar{t}}(\mu, v, h) := (\hat{\mu}, \hat{v}, \hat{h})$.

Transformation LEFT \mathcal{L} : Let (μ, v, h) be PVHA. Let \bar{t} be a time such that: $h_t^+ = 0$ on the interval $[\bar{t} - \Delta t + 2\Delta t^2, \bar{t} + \Delta t^2]$; $h_t^- = 0$ on the interval $[\bar{t} - \Delta t + 2\Delta t^2, \bar{t}]$; $v_t = 0$ on the interval $[\bar{t}, \bar{t} + \Delta t^2]$. Then replace v with \hat{v} defined as follows:

$$\hat{v}_t := \begin{cases} v_t & \text{for } t \in [0, \bar{t} - \Delta t + 2\Delta t^2] \cup (\bar{t} + \Delta t^2, 1], \\ 0 & \text{for } t \in (\bar{t} - \Delta t + 2\Delta t^2, \bar{t} - \Delta t + 3\Delta t^2], \\ v_{t - \Delta t^2} & \text{for } t \in (\bar{t} - \Delta t + 3\Delta t^2, \bar{t} + \Delta t^2]. \end{cases}$$

Also replace h^- with \hat{h}^- defined as follows:

$$\hat{h}_t^- := \begin{cases} h_t^- & \text{for } t \in [0, \bar{t} - \Delta t + 2\Delta t^2] \cup (\bar{t} + \Delta t^2, 1], \\ (\Phi_{[\bar{t} - \Delta t + 2\Delta t^2, \bar{t}]}^{-1} \# h_{t + \Delta t - 2\Delta t^2}^-) & \text{for } t \in (\bar{t} - \Delta t + 2\Delta t^2, \bar{t} - \Delta t + 3\Delta t^2], \\ 0 & \text{for } t \in (\bar{t} - \Delta t + 3\Delta t^2, \bar{t} + \Delta t^2], \end{cases}$$

where $\Phi_{[t_1, t_2]}$ is the flow generated by v .

Keep h^+ . Then the solution $\hat{\mu}$ of (1) with velocity \hat{v} and source $\hat{h} := h^+ - \hat{h}^-$ satisfies

$$\hat{\mu}_t := \begin{cases} \mu_t & \text{for } t \in [0, \bar{t} - \Delta t + 2\Delta t^2] \cup (\bar{t} + \Delta t^2, 1], \\ \mu_{\bar{t} - \Delta t + 2\Delta t^2} - (\Phi_{[\bar{t} - \Delta t + 2\Delta t^2, \bar{t}]}^{-1} \# \left(\int_{\bar{t}}^{t + \Delta t - 2\Delta t^2} ds h_s^- \right)) & \text{for } t \in (\bar{t} - \Delta t + 2\Delta t^2, \bar{t} - \Delta t + 3\Delta t^2], \\ \Phi_{[\bar{t} - \Delta t + 2\Delta t^2, t - \Delta t^2]} \# \hat{\mu}_{\bar{t} - \Delta t + 3\Delta t^2} & \text{for } t \in (\bar{t} - \Delta t + 3\Delta t^2, \bar{t} + \Delta t^2]. \end{cases} \quad (24)$$

We use the notation $\mathcal{L}_{\bar{t}}$ for this transformation applied at a certain time \bar{t} , i.e. $\mathcal{L}_{\bar{t}}(\mu) := \hat{\mu}$ for the transformation of the measure and $\mathcal{L}_{\bar{t}}(\mu, v, h) := (\hat{\mu}, \hat{v}, \hat{h})$.

Transformation RIGHT \mathcal{R} : Let (μ, v, h) be PVHA. Let \bar{t} be a time such that: $h_t^- = 0$ on the interval $[\bar{t} - \Delta t^2, \bar{t} + \Delta t - 2\Delta t^2]$; $v_t = 0$ on the interval $[\bar{t} - \Delta t^2, \bar{t}]$; $h_t^+ = 0$ on the interval $[\bar{t}, \bar{t} + \Delta t - 2\Delta t^2]$. Then replace v with \hat{v} defined as follows:

$$\hat{v}_t := \begin{cases} v_t & \text{for } t \in [0, \bar{t} - \Delta t^2] \cup (\bar{t} + \Delta t - 2\Delta t^2, 1], \\ v_{t+\Delta t^2} & \text{for } t \in (\bar{t} - \Delta t^2, \bar{t} + \Delta t - 3\Delta t^2], \\ 0 & \text{for } t \in (\bar{t} + \Delta t - 3\Delta t^2, \bar{t} + \Delta t - 2\Delta t^2]. \end{cases}$$

Also replace h^+ with \hat{h}^+ defined as follows:

$$\hat{h}_t^+ := \begin{cases} h_t^+ & \text{for } t \in [0, \bar{t} - \Delta t^2] \cup (\bar{t} + \Delta t - 2\Delta t^2, 1], \\ 0 & \text{for } t \in (\bar{t} - \Delta t^2, \bar{t} + \Delta t - 3\Delta t^2], \\ \Phi_{[\bar{t}, \bar{t} + \Delta t - 2\Delta t^2]} \# h_{t-\Delta t+2\Delta t^2}^+ & \text{for } t \in (\bar{t} + \Delta t - 3\Delta t^2, \bar{t} + \Delta t - 2\Delta t^2]. \end{cases}$$

Keep h^- . Then the solution $\hat{\mu}$ of (1) with velocity \hat{v} and source $\hat{h} := \hat{h}^+ - h^-$ satisfies

$$\hat{\mu}_t := \begin{cases} \mu_t & \text{for } t \in [0, \bar{t} - \Delta t^2] \cup (\bar{t} + \Delta t - 2\Delta t^2, 1], \\ \Phi_{[\bar{t}, t+\Delta t^2]} \# \mu_{\bar{t}-\Delta t^2} & \text{for } t \in (\bar{t} - \Delta t^2, \bar{t} + \Delta t - 3\Delta t^2], \\ \hat{\mu}_{\bar{t}+\Delta t-3\Delta t^2} + \Phi_{[\bar{t}, \bar{t}+\Delta t-2\Delta t^2]} \# \left(\int_{\bar{t}-\Delta t^2}^{t-\Delta t+2\Delta t^2} ds h_s^+ \right) & \text{for } t \in (\bar{t} + \Delta t - 3\Delta t^2, \bar{t} + \Delta t - 2\Delta t^2]. \end{cases}$$

We use the notation $\mathcal{R}_{\bar{t}}$ for this transformation applied at a certain time \bar{t} , i.e. $\mathcal{R}_{\bar{t}}(\mu) := \hat{\mu}$ for the transformation of the measure and $\mathcal{R}_{\bar{t}}(\mu, v, h) := (\hat{\mu}, \hat{v}, \hat{h})$.

We now fix the notation for composition of transformations of the kind $\mathcal{D}_{\bar{t}}, \mathcal{L}_{\bar{t}}, \mathcal{R}_{\bar{t}}$. First, define \mathcal{D} as the composition $\mathcal{D} := \mathcal{D}_{\bar{t}_n} \circ \mathcal{D}_{\bar{t}_{n-1}} \circ \dots \circ \mathcal{D}_{\bar{t}_2} \circ \mathcal{D}_{\bar{t}_1}$ where $\bar{t}_1 < \bar{t}_2 < \dots < \bar{t}_n$ are all times in the set $\{0, \Delta t^2, 2\Delta t^2, \dots, (2^{2k} - 1)\Delta t^2, 1\}$ such that $\mathcal{D}_{\bar{t}}$ can be applied. We define \mathcal{L}, \mathcal{R} similarly. Finally, we define $\mathcal{R}\mathcal{L}\mathcal{D}$ as the composition $\mathcal{R} \circ \mathcal{L} \circ \mathcal{D}$.

We now apply $\mathcal{D}, \mathcal{L}, \mathcal{R}$ to $\mu^{[k]}$. One can easily prove that $\mathcal{D}_{\bar{t}}$ can be applied to $\mu^{[k]}$ for $\bar{t} = \Delta t, 2\Delta t, \dots, (2^{-k} - 1)\Delta t$, while one always has $\mathcal{L}(\mu^{[k]}) = \mu^{[k]}$ and $\mathcal{R}(\mu^{[k]}) = \mu^{[k]}$. We apply $\mathcal{R}\mathcal{L}\mathcal{D}$ iteratively to $\mu^{[k]}$. One can observe that, after $2^k - 1$ iterations, the result is a fixed point for $\mathcal{R}\mathcal{L}\mathcal{D}$, i.e. $\mathcal{R}\mathcal{L}\mathcal{D}(\mathcal{R}\mathcal{L}\mathcal{D}^{(2^k-1)}(\mu^{[k]})) = \mathcal{R}\mathcal{L}\mathcal{D}^{(2^k-1)}(\mu^{[k]})$. We define $\tilde{\mu}^{[k]} := \mathcal{R}\mathcal{L}\mathcal{D}^{(2^k-1)}(\mu^{[k]})$ such fixed point.

One can observe that $\tilde{\mu}^{[k]}$ is solution of (1) for a certain \tilde{v}^k, \tilde{h}^k (depending on v^k, h^k) of this kind:

$$\tilde{v}_t^k = 0 \quad \text{for } t \in [0, \Delta t] \cup (1 - \Delta t, 1], \quad \tilde{h}_t^k = \begin{cases} -(\tilde{h}_t^k)^- & \text{for } t \in [0, \Delta t], \\ 0 & \text{for } t \in (\Delta t, 1 - \Delta t], \\ (\tilde{h}_t^k)^+ & \text{for } t \in (1 - \Delta t, 1]. \end{cases}$$

As a consequence, we have

$$\tilde{\mu}_t^{[k]} = \begin{cases} \mu_0 - \int_0^t d\tau (\tilde{h}_\tau^k)^- & \text{for } t \in [0, \Delta t], \\ \tilde{\Phi}_{[\Delta t, t]}^k \# \tilde{\mu}_{\Delta t}^{[k]} & \text{for } t \in (\Delta t, 1 - \Delta t], \\ \tilde{\mu}_{1-\Delta t}^{[k]} + \int_{1-\Delta t}^t d\tau (\tilde{h}_\tau^k)^+ & \text{for } t \in (1 - \Delta t, 1], \end{cases} \quad (25)$$

where $\tilde{\Phi}^k$ is the flow generated by the vector field \tilde{v}^k .

Part 2.2: We now prove three properties of $\tilde{\mu}^{[k]}$.

1. $\tilde{\mu}^{[k]}$ drives μ_0 to $\mu_1^{[k]}$, i.e. $(\tilde{\mu}^{[k]}, \tilde{v}^k, \tilde{h}^k) \in V(\mu_0, \mu_1^{[k]})$. One can prove that, if $\bar{\mu} \in V(\mu_0, \mu_1^{[k]})$, then both $\mathcal{D}_{\bar{t}}(\bar{\mu}), \mathcal{L}_{\bar{t}}(\bar{\mu}), \mathcal{R}_{\bar{t}}(\bar{\mu}) \in V(\mu_0, \mu_1^{[k]})$, since $\bar{\mu}$ is not changed at times 0, 1. Since $\mathcal{R}\mathcal{L}\mathcal{D}^{(2^k-1)}$ is a finite composition of operators $\mathcal{D}_{\bar{t}}, \mathcal{L}_{\bar{t}}, \mathcal{R}_{\bar{t}}$, then $\tilde{\mu}_0^{[k]} = \mu_0, \tilde{\mu}_1^{[k]} = \mu_1^{[k]}$, hence $(\tilde{\mu}^{[k]}, \tilde{v}^k, \tilde{h}^k) \in V(\mu_0, \mu_1^{[k]})$.

2. It holds $\mathcal{B}^{a,b} [\tilde{\mu}^{[k]}, \tilde{v}^k, \tilde{h}^k] \leq \mathcal{B}^{a,b} [\mu^{[k]}, v^k, h^k]$. We prove it by proving that, given (μ, v, h) PVHA, it holds

$$\mathcal{B}^{a,b} [\mathcal{D}_{\bar{t}}(\mu, v, h)] = \mathcal{B}^{a,b} [\mu, v, h], \quad \mathcal{B}^{a,b} [\mathcal{L}_{\bar{t}}(\mu, v, h)] \leq \mathcal{B}^{a,b} [\mu, v, h], \quad \mathcal{B}^{a,b} [\mathcal{R}_{\bar{t}}(\mu, v, h)] \leq \mathcal{B}^{a,b} [\mu, v, h].$$

The idea of the proof for \mathcal{D} is to observe that adding and then removing mass has the same cost than removing and then adding. For \mathcal{L} instead, observe that removing and then transporting mass is cheaper than transporting (a bigger mass) and then removing. For \mathcal{D} , observe that transporting and then adding is cheaper than adding and then transporting.

We first prove $\mathcal{B}^{a,b} [\mu, v, h] - \mathcal{B}^{a,b} [\hat{\mu}, \hat{v}, \hat{h}] = 0$, with $(\hat{\mu}, \hat{v}, \hat{h}) := \mathcal{D}_{\bar{t}}(\mu, v, h)$. Observe that $\int_{\bar{t}-\Delta t^2}^{\bar{t}+\Delta t^2} dt \left(\int_{\mathbb{R}^d} d|\hat{h}_t| \right) = \int_{\bar{t}-\Delta t^2}^{\bar{t}+\Delta t^2} dt \left(\int_{\mathbb{R}^d} d|h_t| \right)$, then $\int_0^1 dt \left(\int_{\mathbb{R}^d} d|\hat{h}_t| \right) = \int_0^1 dt \left(\int_{\mathbb{R}^d} d|h_t| \right)$. We then have the result.

We now prove $\mathcal{B}^{a,b} [\mu, v, h] - \mathcal{B}^{a,b} [\hat{\mu}, \hat{v}, \hat{h}] \geq 0$, with $(\hat{\mu}, \hat{v}, \hat{h}) := \mathcal{L}_{\bar{t}}(\mu, v, h)$. Denote with where $\Phi_{[t_1, t_2]}^k$ the flow generated by v . First observe that

$$\begin{aligned} \int_{\bar{t}-\Delta t+2\Delta t^2}^{\bar{t}+\Delta t^2} dt \left(\int_{\mathbb{R}^d} d|\hat{h}_t| \right) &= \int_{\bar{t}-\Delta t+2\Delta t^2}^{\bar{t}-\Delta t+3\Delta t^2} dt \left(\int_{\mathbb{R}^d} d \left| (\Phi_{[\bar{t}-\Delta t+2\Delta t^2, \bar{t}]}^{-1})^\# h_{t+\Delta t-2\Delta t^2}^- \right| \right) = \\ &= \int_{\bar{t}}^{\bar{t}+\Delta t^2} dt \left(\int_{\mathbb{R}^d} dh_t^- \right) = \int_{\bar{t}-\Delta t+2\Delta t^2}^{\bar{t}+\Delta t^2} dt \left(\int_{\mathbb{R}^d} d|h_t| \right), \end{aligned} \quad (26)$$

where we have used that mass is conserved by the push-forward of measures $\int_{\mathbb{R}^d} d|\mu| = \int_{\mathbb{R}^d} d|\phi\#\mu|$. As a consequence, it holds $\int_0^1 dt \left(\int_{\mathbb{R}^d} d|\hat{h}_t| \right) = \int_0^1 dt \left(\int_{\mathbb{R}^d} d|h_t| \right)$.

Now observe that, given two positive masses μ, ν satisfying $\mu \leq \nu$, then $\int_{\mathbb{R}^d} f d\mu \leq \int_{\mathbb{R}^d} f d\nu$ for f positive function. Using the definition of $\hat{\mu}$ in (24), we have $\hat{\mu}_{\bar{t}-\Delta t+3\Delta t^2} \leq \mu_{\bar{t}-\Delta t+2\Delta t^2}$, that also implies for $t \in [\bar{t}-\Delta t+3\Delta t^2, \bar{t}+\Delta t^2]$

$$\hat{\mu}_t = \Phi_{[\bar{t}-\Delta t+2\Delta t^2, t-\Delta t^2]}^\# \hat{\mu}_{\bar{t}-\Delta t+3\Delta t^2} \leq \Phi_{[\bar{t}-\Delta t+2\Delta t^2, t-\Delta t^2]}^\# \mu_{\bar{t}-\Delta t+2\Delta t^2} = \mu_{t-\Delta t^2}.$$

Also recalling that $\hat{v}_t = v_{t-\Delta t}$ for $t \in [\bar{t}-\Delta t+3\Delta t^2, \bar{t}+\Delta t^2]$, we have

$$\int_{\bar{t}-\Delta t+3\Delta t^2}^{\bar{t}+\Delta t^2} dt \left(\int_{\mathbb{R}^d} d\hat{\mu}_t |\hat{v}_t|^2 \right) \leq \int_{\bar{t}-\Delta t+3\Delta t^2}^{\bar{t}+\Delta t^2} dt \left(\int_{\mathbb{R}^d} d\mu_{t-\Delta t^2} |v_{t-\Delta t}|^2 \right) = \int_{\bar{t}-\Delta t+2\Delta t^2}^{\bar{t}} dt \left(\int_{\mathbb{R}^d} d\mu_t |v_t|^2 \right).$$

By observing that $\hat{v}_t = 0$ for $t \in (\bar{t}-\Delta t+2\Delta t^2, \bar{t}-\Delta t+3\Delta t^2]$ and $v_t = 0$ for $t \in (\bar{t}, \bar{t}+\Delta t^2]$, we also have

$$\int_{\bar{t}-\Delta t+2\Delta t^2}^{\bar{t}+\Delta t^2} dt \left(\int_{\mathbb{R}^d} d\hat{\mu}_t |\hat{v}_t|^2 \right) \leq \int_{\bar{t}-\Delta t+2\Delta t^2}^{\bar{t}+\Delta t^2} dt \left(\int_{\mathbb{R}^d} d\mu_t |v_t|^2 \right). \quad (27)$$

Since $(\mu_t, v_t) = (\hat{\mu}_t, \hat{v}_t)$ for $t \in [0, \bar{t}-\Delta t+2\Delta t^2] \cup (\bar{t}+\Delta t^2, 1]$, then estimates (26) and (27) give the result $\mathcal{B}^{a,b} [\mu, v, h] \geq \mathcal{B}^{a,b} [\mathcal{L}(\mu, v, h)]$.

The proof for \mathcal{D} is equivalent to the proof for \mathcal{L} .

3. It holds $T_2^{a,b} (\mu_0, \mu_1^{[k]}) \leq \mathcal{B}^{a,b} [\tilde{\mu}^{[k]}, \tilde{v}^k, \tilde{h}^k]$. Observing the explicit structure of $\tilde{\mu}^{[k]}$ given in (25) one can observe that $\tilde{\mu}_{\Delta t}^{[k]} \leq \tilde{\mu}_0^{[k]} = \mu_0$, $\tilde{\mu}_{1-\Delta t}^{[k]} \leq \tilde{\mu}_1^{[k]} = \mu_1^{[k]}$ and that $\tilde{\mu}_{1-\Delta t}^{[k]} = \tilde{\Phi}_{[\Delta t, 1-\Delta t]}^k \# \tilde{\mu}_{\Delta t}^{[k]}$. As a consequence, one has

$$\begin{aligned} T_2^{a,b} (\mu_0, \tilde{\mu}_1^{[k]}) &= \inf_{m_0, m_1 \in \mathcal{M}_{0^c}^{a,b}} a^2 \left(|\mu_0 - m_0| + |\mu_1^{[k]} - m_1| \right)^2 + b^2 W_2^2(m_0, m_1) \leq \\ &\leq a^2 \left(|\tilde{\mu}_0^{[k]} - \tilde{\mu}_{\Delta t}^{[k]}| + |\tilde{\mu}_1^{[k]} - \tilde{\mu}_{1-\Delta t}^{[k]}| \right)^2 + b^2 W_2^2(\tilde{\mu}_{\Delta t}^{[k]}, \tilde{\mu}_{1-\Delta t}^{[k]}) = \\ &= a^2 \left(\int_0^{\Delta t} dt \left(\int_{\mathbb{R}^d} d(\tilde{h}_t^k)^- \right) + \int_{1-\Delta t}^1 dt \left(\int_{\mathbb{R}^d} d(\tilde{h}_t^k)^+ \right) \right)^2 + b^2 W_2^2(\tilde{\mu}_{\Delta t}^{[k]}, \tilde{\Phi}_{[\Delta t, 1-\Delta t]}^k \# \tilde{\mu}_{\Delta t}^{[k]}). \end{aligned} \quad (28)$$

In the first inequality we have used the choice $m_0 = \tilde{\mu}_{\Delta t}^{[k]}, m_1 = \tilde{\mu}_{1-\Delta t}^{[k]}$ to estimate the infimum. We now observe that the first term of the right hand side of (28) coincides with $a^2 \left(\int_0^1 dt \left(\int_{\mathbb{R}^d} d|\tilde{h}_t^k| \right) \right)^2$. For the second term, introduce the vector field $\tilde{v}_t^k := (1 - 2\Delta t)\tilde{v}_{(1-2\Delta t)t+\Delta t}^k$ and observe that the corresponding flow $\tilde{\Phi}^k$ satisfies $\tilde{\Phi}_{[0,1]}^k \# \tilde{\mu}_{\Delta t}^{[k]} = \tilde{\mu}_{1-\Delta t}^{[k]}$. Using the standard Benamou-Brenier formula (7), it holds

$$\begin{aligned} W_2^2 \left(\tilde{\mu}_{\Delta t}^{[k]}, \tilde{\Phi}_{[\Delta t, 1-\Delta t]}^k \# \tilde{\mu}_{\Delta t}^{[k]} \right) &\leq \int_0^1 d\tau \left(\int_{\mathbb{R}^d} d(\tilde{\Phi}_{[0,\tau]}^k \# \tilde{\mu}_{\Delta t}^{[k]}) |\tilde{v}_\tau^k|^2 \right) = \\ &= \int_{\Delta t}^{1-\Delta t} (1 - 2\Delta t)^{-1} dt \left(\int_{\mathbb{R}^d} d(\tilde{\Phi}_{[\Delta t, t]}^k \# \tilde{\mu}_{\Delta t}^{[k]}) (1 - 2\Delta t)^2 |\tilde{v}_t^k|^2 \right), \end{aligned}$$

where we have applied the change of variable $\tau \rightarrow t = (1 - 2\Delta t)\tau + \Delta t$. Observing that $\tilde{v}_t^k = 0$ for $t \in [0, \Delta t] \cup (1 - \Delta t, 1]$, and that $\tilde{\Phi}_{[\Delta t, t]}^k \# \tilde{\mu}_{\Delta t}^{[k]} = \tilde{\mu}_t^{[k]}$, we have

$$W_2^2 \left(\tilde{\mu}_{\Delta t}^{[k]}, \tilde{\Phi}_{[\Delta t, 1-\Delta t]}^k \# \tilde{\mu}_{\Delta t}^{[k]} \right) \leq (1 - 2\Delta t) \int_0^1 dt \left(\int_{\mathbb{R}^d} d\tilde{\mu}_t^{[k]} |\tilde{v}_t^k|^2 \right) \leq \int_0^1 dt \left(\int_{\mathbb{R}^d} d\tilde{\mu}_t^{[k]} |\tilde{v}_t^k|^2 \right).$$

Going back to (28), we have

$$T_2^{a,b}(\mu_0, \mu_1) \leq a^2 \left(\int_0^1 dt \left(\int_{\mathbb{R}^d} d|\tilde{h}_t^k| \right) \right)^2 + b^2 \int_0^1 dt \left(\int_{\mathbb{R}^d} d\tilde{\mu}_t^{[k]} |\tilde{v}_t^k|^2 \right) = \mathcal{B}^{a,b} [\tilde{\mu}^{[k]}, \tilde{v}^k, \tilde{h}^k].$$

Part 2.3: We now prove (12). For each k it holds $T_2^{a,b}(\mu_0, \mu_1^{[k]}) \leq \mathcal{B}^{a,b} [\tilde{\mu}^{[k]}, \tilde{v}^k, \tilde{h}^k] \leq \mathcal{B}^{a,b} [\mu^{[k]}, v^k, h^k]$. Since $\lim_k |W_2^{a,b}(\mu_0, \mu_1^{[k]}) - W_2^{a,b}(\mu_0, \mu_1)| \leq \lim_k W_2^{a,b}(\mu_1^{[k]}, \mu_1) \leq \lim_k d(\mu^{[k]}, \mu) = 0$, then $\lim_k T_2^{a,b}(\mu_0, \mu_1^{[k]}) = T_2^{a,b}(\mu_0, \mu_1)$. Then

$$T_2^{a,b}(\mu_0, \mu_1) = \lim_k T_2^{a,b}(\mu_0, \mu_1^{[k]}) \leq \lim_k \mathcal{B}^{a,b} [\mu^{[k]}, v^k, h^k] = \mathcal{B}^{a,b} [\mu, v, h].$$

□

5 Proof of the generalized Benamou-Brenier formula

In this section we prove Theorem 10. We divide the proof in two parts. In part 1, we generalize the inequality $T_2^{a,b}(\mu_0, \mu_1) \leq \mathcal{B}^{a,b}[\mu, v, h]$. In Part 2, we prove the converse inequality.

Part 1: We prove that $T_2^{a,b}(\mu_0, \mu_1) \leq \mathcal{B}^{a,b}[\mu, v, h]$. We observe that we have removed hypotheses with respect to Lemma 12. For h , we have passed from L^∞ to L^1 regularity. On the side of v , we have passed from Lipschitz continuity with respect to space and uniform boundedness to $v_t \in L^2(dt d\mu_t)$. The generalization for the term h is provided in Part 1.1, while the subsequent generalization for the term v is provided in Part 1.2.

Part 1.1: We first prove that one can pass from the case of h in L^∞ to the case of h in L^1 . The idea is to define $\mu^{[k]}$ as in the proof of Lemma 12, Part 1, and to provide similar estimates. First define

$$p_n^k := \int_{n2^{-k}}^{(n+1)2^{-k}} dt |h_t|.$$

Now repeat Part 1 of the proof of Lemma 12. First observe that all measures have bounded mass, bounded by $m := |\mu_0| + \int_0^1 dt |h_t|$. Then replace (13) and (14) by

$$W_2^{a,b} \left(\mu_{n\Delta t}^{[k]}, \mu_{n\Delta t+\tau}^{[k]} \right) \leq \begin{cases} a |H_{[n\Delta t, n\Delta t+\tau]}^k| \leq ap_n^k & \text{for } \tau \in (0, \Delta t^2], \\ b \frac{(\tau - \Delta t^2)\Delta t}{\Delta t - 2\Delta t^2} Mm + ap_n^k & \text{for } \tau \in (\Delta t^2, \Delta t - \Delta t^2], \\ b\Delta t Mm + 2ap_n^k & \text{for } \tau \in (\Delta t - \Delta t^2, \Delta t], \end{cases} \quad (29)$$

and

$$W_2^{a,b} \left(\mu_{n\Delta t}^{[k]}, \mu_{n\Delta t+\tau}^{[k]} \right) \leq 2ap_n^k + bMm\Delta t. \quad (30)$$

With the same computations, replace (15) by

$$W_2^{a,b} \left(\mu_{(n+2)\Delta t}^{[k-1]}, \mu_{(n+2)\Delta t}^{[k]} \right) \leq e^{\frac{3}{2}L\Delta t} W_2^{a,b} \left(\mu_{n\Delta t}^{[k-1]}, \mu_{n\Delta t}^{[k]} \right) + be^{\frac{3}{2}L\Delta t} \Delta t M p_n^k + b\Delta t M p_n^k.$$

Observe that, given a sequence $a_{n+1} \leq Ca_n + b_n$ with $a_0 = 0$ and $C > 1$, one can estimate $a_n \leq C^{n-1}b_0 + C^{n-2}b_1 + \dots + b_{n-1} \leq C^{n-1} \sum_{i=1}^{n-1} b_i$. In our case, we have the estimate $W_2^{a,b} \left(\mu_{(n+2)\Delta t}^{[k-1]}, \mu_{(n+2)\Delta t}^{[k]} \right) \leq e^{\frac{3}{2}L\Delta t} (b\Delta t M) (e^{\frac{3}{2}L\Delta t} + 1) \sum_{i=0}^n p_i^k$. Then, using the fact that $n\Delta t \leq 1$ and $\sum_{i=0}^n p_i^k \leq \sum_{i=0}^{2^k-1} p_n^k = \int_0^1 dt |h_t|$, one has the following estimate, independent on n :

$$W_2^{a,b} \left(\mu_{n\Delta t}^{[k-1]}, \mu_{n\Delta t}^{[k]} \right) \leq 3bMe^{\frac{3}{4}L} \left(\int_0^1 dt |h_t| \right) 2^{-k}.$$

It is then easy to prove $W_2^{a,b} \left(\mu_{n2^{-k}}^{[k]}, \mu_{n2^{-k}}^{[k+l]} \right) \leq 6bMe^{\frac{3}{4}L} \left(\int_0^1 dt |h_t| \right) 2^{-k}$. Using (30) and choosing n even such that $n2^{-k} \leq t < (n+2)2^{-k}$, estimate

$$\begin{aligned} W_2^{a,b} \left(\mu_t^{[k]}, \mu_t^{[k+l]} \right) &\leq 2a(p_n^k + p_{n+1}^k) + 2bMm2^{-k} + 6bMe^{\frac{3}{4}L} \left(\int_0^1 dt |h_t| \right) 2^{-k} + 2a \sum_{i=n2^l}^{(n+1)2^l} p_i^{k+l} + 2^{l+1}bMm2^{-k-l} \leq \\ &\leq 2a \int_{n2^{-k}}^{(n+2)2^{-k}} |h_t| dt + C_2 2^{-k} + 2a \int_{n2^{-k}}^{(n+2)2^{-k}} |h_t| dt, \end{aligned}$$

with $C_2 := 4bMm + 6bMe^{\frac{3}{4}L} \left(\int_0^1 dt |h_t| \right)$. Since $|h_t| \in L^1([0, 1], R)$, for each ε it exists $\psi(\varepsilon) := \sup_{t \in [0, 1-\varepsilon]} \int_t^{t+\varepsilon} |h_t|$ and it satisfies $\psi(\varepsilon) \rightarrow 0$ for $\varepsilon \rightarrow 0$. Then $d(\mu^{[k]}, \mu^{[k+l]}) \leq 4a\psi(2^{-k+1}) + 2^{-k}$, hence $\mu^{[k]}$ is a Cauchy sequence in $C([0, 1], \mathcal{M})$.

We are now left to prove that the limit $\mu^* = \lim_k \mu^{[k]}$ coincides with μ . We prove it by proving that it is a solution of (1). Perform the same estimates as in the proof of Lemma 12, Part 1.3, with the following modifications: For the limit 2:

1. In estimate (19), estimate $W_2^{a,b} \left(\mu_{n\Delta t+t}^{[k]}, \mu_{n\Delta t+\Delta t^2}^{[k]} \right) \leq \Delta t^{-1} p_n^k$;
2. Similarly, in estimate (20), estimate $W_2^{a,b} \left(\mu_{n\Delta t+t}^{[k]}, \mu_{n\Delta t+\Delta t-\Delta t^2}^{[k]} \right) \leq \Delta t^{-1} p_n^k$;
3. This induces a change in (21), that can be estimated with $Cd(\mu^*, \mu^{[k]})\Delta t + Cp_n^k\Delta t + 3C\Delta t^3$;
4. This in turn induces the following change in the main estimate

$$\begin{aligned} \lim_k \left| \int_0^1 dt \left(\int_{\mathbb{R}^d} d\mu_t^* v_t \cdot \nabla f_t - d\mu_t^{[k]} v_t^k \cdot \nabla f_t \right) \right| &\leq \lim_k \left(Cd(\mu^*, \mu^{[k]}) + C2^{-k} \sum_0^{2^k-1} p_n^k + 3C2^{-2k} \right) \leq \\ &\leq 0 + C \lim_k 2^{-k} \int_0^1 |h_t| dt + 0 = 0. \end{aligned}$$

For the limit 3:

1. Estimate (22) with $\int_0^{\Delta t} dt |h_{n\Delta t+t}| \|f_t\|_{Lip}(t - t\Delta t) \leq \Delta t \int_0^{\Delta t} dt |h_{n\Delta t+t}| \|f_t\|_{Lip}$.

2. This in turn induces the following change in the main estimate

$$\lim_k \left| \int_0^1 dt \left(\int_{\mathbb{R}^d} d(h_t - h_t^k) f_t \right) \right| \leq \lim_k 2^{-k} \|f_t\|_{Lip} \int_0^1 dt |h_{n\Delta t+t}| = 0.$$

After having proved that μ^* is a solution, one has $\mu^* = \mu$ by observing that existence and uniqueness is guaranteed for $h \in L^1([0, 1], \mathcal{M}_0^{ac})$ too.

Finally, one can easily prove $\mathcal{B}^{a,b}[\mu^k, v^k, h^k] \rightarrow \mathcal{B}^{a,b}[\mu, v, h]$ by following the estimates of Lemma 12, Part 1.4. The only difference is in the estimate of the second term of (23), in which one has to replace the use of (14) by (30). This proves that Lemma 12 holds even with $h \in L^1([0, 1], \mathcal{M}_0^{ac})$.

Part 1.2: We now generalize Lemma 12 to $v \in L^2(dt d\mu_t)$. The proof is a generalization of the proof of the Benamou-Brenier formula given in [7, Theorem 8.1], Step 2. The main idea is to introduce the variable $m_t := \rho_t v_t$, where ρ_t is the density of μ_t , and observe that $\rho_t |v_t|^2 = |m_t|^2 / \rho_t$ is a convex function of ρ_t, m_t . Then, we write $\mathcal{B}^{a,b}[\mu, m, h]$ with an abuse of notation, and observe that it is convex with respect to its arguments. The presence of the term h makes no difference on this point with respect to [7, Theorem 8.1], Step 2.

Let (μ, v, h) satisfy hypotheses of Theorem 10. Denote with Ω an open bounded set containing the support of all μ_t , for $t \in [0, 1]$. As already discussed in Remark 9, we also have that Ω contains the support of all h_t , for $t \in [0, 1]$. Define a measure $\bar{\mu}_0 \in \mathcal{M}_0^{ac}$ such that its density $\bar{\rho}_0$ is a function in $C_c^\infty(\mathbb{R}^d)$ and satisfies $\bar{\rho}_0 \geq 1$ in Ω . Define now $\bar{\mu}$ as the solution of $\partial_t \bar{\mu} = h$, with initial datum $\bar{\mu}_0 := \bar{\mu}_0 + \int_0^1 dt h_t^-$. It is easy to observe that $\bar{\mu}_0 \leq \bar{\mu}_t$ for all $t \in [0, 1]$, then $\bar{\mu}_t$ has a density $\bar{\rho}_t$ satisfying $\bar{\rho}_0 \geq 1$ in Ω , and $\bar{\mu}_t$ has a bounded support. Moreover, $|\bar{\mu}_t| \leq |\bar{\mu}_0| + 2 \int_0^1 dt |h_t|$, then $\bar{\mu}_t$ has bounded mass.

For each $\lambda \in [0, 1]$, define $\mu_t^\lambda := (1 - \lambda)\mu_t + \lambda \bar{\mu}_t$ and $m_t^\lambda := (1 - \lambda)m_t$. Observe that for each λ it holds $\partial_t \mu_t^\lambda + \nabla \cdot m_t^\lambda = h$. Denote with ρ_t^λ the density of μ_t^λ . For simplicity of notation, fix now a certain $\lambda \in (0, 1)$ and denote $\eta := \mu^\lambda$, with density χ , and $n := m^\lambda$. Observe that the support of n is contained in Ω and that χ is uniformly bounded from below on a neighborhood of Ω .

Define now, for each $\varepsilon > 0$, a regularizing kernel $r_\varepsilon(t, x) := \frac{1}{\varepsilon^{d+1}} r_1(x/\varepsilon) r_2(t/\varepsilon)$, where $r_1 \in C_c^\infty(\mathbb{R}^d)$, $r_2 \in C_c^\infty(\mathbb{R}^+)$ satisfy $r_i \geq 0$ and $\text{supp}(r_1) \in B_1(0)$, $\text{supp}(r_2) \in (0, 1)$. Define $\chi^\varepsilon := \chi * r_\varepsilon$, $n^\varepsilon := n * r_\varepsilon$ and $h^\varepsilon := h * r_\varepsilon$. Denote with η_t^ε the measure with density χ_t^ε . It is clear that $\partial_t \eta^\varepsilon + \nabla \cdot n^\varepsilon = h^\varepsilon$ for $t \in (\varepsilon, 1 - \varepsilon)$. Observe now that, for a sufficiently small ε , it holds that χ is uniformly bounded from below on a neighborhood of the support of n^ε . For this reason, one can define $v_t^\varepsilon := n_t^\varepsilon / \chi_t^\varepsilon$, that is well-defined⁵ for all $t \in (\varepsilon, 1 - \varepsilon)$. Moreover, v^ε is a C^1 function, with compact support. Then, using the results of Step 1.1 and time reparametrization, one has

$$\mathcal{B}^{a,b}[\eta^\varepsilon, n^\varepsilon, h^\varepsilon] = \mathcal{B}^{a,b}[\eta^\varepsilon, v^\varepsilon, h^\varepsilon] \geq (1 - 2\varepsilon)^{-1} T_2^{a,b}(\eta^\varepsilon, \eta_{1-\varepsilon}^\varepsilon) \geq T_2^{a,b}(\eta^\varepsilon, \eta_{1-\varepsilon}^\varepsilon).$$

Observe now that, again by convexity of $|n|^2/\chi$ with respect to its arguments, one has

$$\chi_t^\varepsilon |v_t^\varepsilon|^2 = \frac{|n_t^\varepsilon|^2}{\chi_t * r_\varepsilon} = \frac{|n_t * r_\varepsilon|^2}{\chi_t * r_\varepsilon} \leq \frac{|n_t|^2}{\chi_t} * r_\varepsilon,$$

and that

$$\int_\varepsilon^{1-\varepsilon} dt \int_{\mathbb{R}^d} d|h_t^\varepsilon| = \left(\int_0^1 dt \int_{\mathbb{R}^d} d|h_t| \right) \left(\int_0^1 dt \int_{\mathbb{R}^d} dx r_\varepsilon(t, x) \right) = \int_0^1 dt \int_{\mathbb{R}^d} d|h_t|.$$

This implies that $\mathcal{B}^{a,b}[\eta, n, h] \geq \mathcal{B}^{a,b}[\eta^\varepsilon, n^\varepsilon, h^\varepsilon] \geq T_2^{a,b}(\eta^\varepsilon, \eta_{1-\varepsilon}^\varepsilon)$.

We now prove that, for $\varepsilon \rightarrow 0$, it holds $T_2^{a,b}(\eta^\varepsilon, \eta_{1-\varepsilon}^\varepsilon) \rightarrow T_2^{a,b}(\eta_0, \eta_1)$. Since $T_2^{a,b}(\mu, \nu) = \left(W_2^{a,b}(\mu, \nu) \right)^2$, then it is equivalent to prove that $W_2^{a,b}(\eta^\varepsilon, \eta_0) \rightarrow 0$ and $W_2^{a,b}(\eta_{1-\varepsilon}^\varepsilon, \eta_1) \rightarrow 0$. First observe that the supports of the η^ε are uniformly bounded, then the sequence η^ε is tight. Observe now that $\eta^\varepsilon \rightharpoonup \eta$ and, due to continuity of both $\eta_t^\varepsilon, \eta_t$ with respect to time, then $\eta^\varepsilon \rightharpoonup \eta_0$. Recall now that $W_2^{a,b}$ metrizes weak convergence

⁵Here it is necessary to fix the convention $0/0 := 0$.

for tight sequences (see Theorem 4), then $W_2^{a,b}(\eta_\varepsilon^\varepsilon, \eta_0) \rightarrow 0$. Similarly, one can prove $W_2^{a,b}(\eta_{1-\varepsilon}^\varepsilon, \eta_1) \rightarrow 0$. This proves $\mathcal{B}^{a,b}[\eta, n, h] \geq T_2^{a,b}(\eta_0, \eta_1)$.

Going back to the notation μ_t^λ , the previous result reads as $\mathcal{B}^{a,b}[\mu^\lambda, m^\lambda, h] \geq T_2^{a,b}(\mu_0^\lambda, \mu_1^\lambda)$ for all $\lambda \in (0, 1)$. Due to convexity of $\mathcal{B}^{a,b}$, one has

$$\begin{aligned} \mathcal{B}^{a,b}[\mu^\lambda, m^\lambda, h] &\leq (1-\lambda)\mathcal{B}^{a,b}[\mu, m, h] + \lambda\mathcal{B}^{a,b}[\bar{\mu}, 0, h] = (1-\lambda) \left(\int_0^1 dt \int_{\mathbb{R}^d} d\mu_t |v_t|^2 + \left(\int_0^1 dt |h_t| \right) \right) + \\ &\quad + \lambda \left(\int_0^1 dt |h_t| \right) \leq \left(\int_0^1 dt \int_{\mathbb{R}^d} d\mu_t |v_t|^2 + \left(\int_0^1 dt |h_t| \right) \right) = \mathcal{B}^{a,b}[\mu, m, h]. \end{aligned}$$

We now prove that $T_2^{a,b}(\mu_0^\lambda, \mu_1^\lambda) \rightarrow T_2^{a,b}(\mu_0, \mu_1)$ for $\lambda \rightarrow 0$. Observe that $W_2^{a,b}(\mu_0^\lambda, \mu_0) \rightarrow 0$ for $\lambda \rightarrow 0$, which can be easily proven by

$$W_2^{a,b}(\mu_0^\lambda, \mu_0) \leq W_2^{a,b}((1-\lambda)\mu_0 + \lambda\bar{\mu}_0, (1-\lambda)\mu_0) + W_2^{a,b}((1-\lambda)\mu_0, \mu_0) \leq \lambda|\bar{\mu}_0| + \lambda|\mu_0|,$$

and by recalling that $|\bar{\mu}_0|$ and $|\mu_0|$ are finite. Similarly, both $|\bar{\mu}_1|$ and $|\mu_1|$ are finite, then $W_2^{a,b}(\mu_1^\lambda, \mu_1) \rightarrow 0$ for $\lambda \rightarrow 0$. Then $W_2^{a,b}(\mu_0^\lambda, \mu_1^\lambda) \rightarrow W_2^{a,b}(\mu_0, \mu_1)$, hence $T_2^{a,b}(\mu_0^\lambda, \mu_1^\lambda) \rightarrow T_2^{a,b}(\mu_0, \mu_1)$.

Summing up, we have

$$T_2^{a,b}(\mu_0, \mu_1) = \lim_{\lambda \rightarrow 0} T_2^{a,b}(\mu_0^\lambda, \mu_1^\lambda) \leq \lim_{\lambda \rightarrow 0} \mathcal{B}^{a,b}[\mu^\lambda, m^\lambda, h] \leq \mathcal{B}^{a,b}[\mu, m, h].$$

Part 2: We now prove that $\inf \{ \mathcal{B}^{a,b}[\mu, v, h] \mid (\mu, v, h) \in V(\mu_0, \mu_1) \} \leq T_2^{a,b}(\mu_0, \mu_1)$ by giving a sequence (μ^k, v^k, h^k) realizing the equality at the limit. First of all, observe that there exists⁶ a choice $\tilde{\mu}_0, \tilde{\mu}_1$ such that

$$T_2^{a,b}(\mu_0, \mu_1) = a^2(|\mu_0 - \tilde{\mu}_0| + |\mu_1 - \tilde{\mu}_1|)^2 + b^2 W_2^2(\tilde{\mu}_0, \tilde{\mu}_1),$$

and with $\tilde{\mu}_0 \leq \mu_0, \tilde{\mu}_1 \leq \mu_1$. Define ψ to be the optimal map realizing $W_2^2(\tilde{\mu}_0, \tilde{\mu}_1)$, that exists since $\tilde{\mu}_0, \tilde{\mu}_1 \in \mathcal{M}_0^{ac}$. Also define (see [7])

$$\psi_t(x) := (1-t)x + t\psi(x), \quad v_t^* := (\psi - \text{Id}) \circ \psi_t^{-1}, \quad \tilde{\mu}_t := \psi_t \# \tilde{\mu}_0,$$

and recall that $(\tilde{\mu}, v^*)$ is the choice realizing the equality in the standard Benamou-Brenier formula (7), i.e.

$$W_2^2(\tilde{\mu}_0, \tilde{\mu}_1) = \mathcal{A}[\tilde{\mu}, v^*].$$

The idea of the proof is to write a dynamics first driving μ_0 to $\tilde{\mu}_0$ via removal of mass, then $\tilde{\mu}_0$ to $\tilde{\mu}_1$ via push-forward of measure, and finally $\tilde{\mu}_1$ to μ_1 with creation of mass.

Fix an integer k and define $\Delta t := 2^{-k}$. We define v^k, h^k as follows:

$$v_t^k := \begin{cases} 0 & \text{for } t \in [0, \Delta t] \cup (1 - \Delta t, 1], \\ (1 - 2\Delta t)^{-1} v_{\frac{t - \Delta t}{1 - 2\Delta t}}^* & \text{for } t \in (\Delta t, 1 - \Delta t], \end{cases} \quad h_t^k := \begin{cases} -\Delta t^{-1}(\mu_0 - \tilde{\mu}_0) & \text{for } t \in [0, \Delta t], \\ 0 & \text{for } t \in (\Delta t, 1 - \Delta t], \\ \Delta t^{-1}(\mu_1 - \tilde{\mu}_1) & \text{for } t \in [1 - \Delta t, 1]. \end{cases}$$

One can prove that the expression of the solution μ^k of (1) with vector field v^k and source h^k satisfies

$$\mu_t^k := \begin{cases} (1 - \Delta t^{-1} t)\mu_0 + \Delta t^{-1} t \tilde{\mu}_0 & \text{for } t \in [0, \Delta t], \\ \tilde{\mu}_{\frac{t - \Delta t}{1 - 2\Delta t}} & \text{for } t \in (\Delta t, 1 - \Delta t], \\ \Delta t^{-1} (t - 1 + \Delta t)\mu_1 + \Delta t^{-1} (1 - t)\tilde{\mu}_1 & \text{for } t \in [1 - \Delta t, 1]. \end{cases}$$

⁶The result can be proven even without assuming the existence of $\tilde{\mu}_0, \tilde{\mu}_1$, via a double limit and a diagonalization argument.

In particular, one has $(\mu^k, v^k, h^k) \in V(\mu_0, \mu_1)$. Similarly to proof of Lemma 12, Part 2.2, Property 3, one can prove that

$$\int_0^1 dt \left(\int_{\mathbb{R}^d} d|h_k| \right) = |\mu_0 - \tilde{\mu}_0| + |\mu_1 - \tilde{\mu}_1|, \quad W_2^2(\tilde{\mu}_0, \tilde{\mu}_1) = \int_0^1 d\tau \left(\int_{\mathbb{R}^d} d\tilde{\mu}_\tau |v_\tau^*|^2 \right) = (1 - 2\Delta t) \int_0^1 dt \left(\int_{\mathbb{R}^d} d\mu_t |v_\tau^k|^2 \right).$$

One then has

$$\mathcal{B}^{a,b}[\mu^k, v^k, h^k] = a^2 (|\mu_0 - \tilde{\mu}_0| + |\mu_1 - \tilde{\mu}_1|)^2 + b^2 (1 - 2\Delta t)^{-1} W_2^2(\tilde{\mu}_0, \tilde{\mu}_1) \leq (1 - 2^{-k+1})^{-1} T_2^{a,b}(\mu_0, \mu_1).$$

Passing to the limit, we have the result

$$\inf \{ \mathcal{B}^{a,b}[\mu, v, h] \mid (\mu, v, h) \in V(\mu_0, \mu_1) \} \leq \lim_k \mathcal{B}^{a,b}[\mu^k, v^k, h^k] \leq T_2^{a,b}(\mu_0, \mu_1).$$

□

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